DORFMAN LECTURE NOTES

PART I:

QUANTUM CHAOS, IHP

PARIS NOVEMBER 2007

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I. Classical Mechanics

Hamiltonian Systems

$$H = \sum^{N} \frac{p_i^2}{2m} + V(q_1, \cdots q_N)$$ typically, but not always (hard spheres, billiards, magnetic field)

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

Poisson bracket $\{f, g\}$

$$\{f, g\} \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

Note

$$\{f, H\} = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \frac{df}{dt} - \frac{\partial f}{\partial t}$$

Note that

a) $\{f, g\} = -\{g, f\}$

b) $\{f, gh\} = \{f, g\}h + g\{f, h\}$

c) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$

$\{,\}$ Acts like a commutator

Constant of motion g not explicit fn of time

$$\{g, H\} = 0$$

If g_i, g_j are const of motion, then

$\{g_i, g_j\}$ is a const of motion too

Proof - use Jacobi

$$\{\{g_i, g_j\}, H\} + \{\overset{0}{\overset{\shortparallel}{\{H, g_i\}}}, g_j\} + \{\overset{0}{\overset{\shortparallel}{\{g_j, H\}}}, g_i\} =$$

$$\{\{g_i, g_j\}, H\} = 0$$

<u>Definition</u>: Completely Integrable System

Suppose we have a phase space $\Gamma = (q_1, \dots q_N, p_1, \dots p_N)$ and a Hamiltonian $H(\Gamma)$, then the system is completely integrable if there are N ~~constants of~~ functions $f_1(\Gamma), \dots, f_N(\Gamma)$ such that

1) $f_1(\Gamma) = H(\Gamma)$
2) $\{f_i, f_j\} = 0$ for all i, j
3) The f_i's are linearly independent, i.e rank of the Jacobian $J(f_1, \dots f_N) = N$

Example: Two body problem with central potential. Reduces to

Center of Mass Motion + Relative motion

3 spatial dimensions

C of M: $\vec{P}_{tot} = \text{const.}$

Relative Motion: Energy = const + two angular momenta P_θ, P_ϕ

Can be written in terms of action-angle variables

$$H = \frac{-\text{const}}{(J_r + J_\theta + J_\phi)^2} \quad \text{for bound orbits}$$

Conjugate angles ~~[illegible]~~
are linear functions of time

$$\dot{\alpha}_i = \frac{\partial H}{\partial J_i}, \quad \alpha_i(t) = \omega_i t + \alpha_i(0)$$

Motion can be described by invariant to

All integrable Systems can be reducted to motion on invariant tori.

Simple Pendulum

GENERAL STRUCTURE:

Generating function $S(\vec{J}, \vec{q})$ such that

$$\vec{\theta} = \frac{\partial S(J, \vec{q})}{\partial \vec{J}}, \quad \vec{P} = \frac{\partial S(J, \vec{q})}{\partial \vec{q}}$$

Then

$$\Delta_i S = \oint_{\gamma_i} \vec{P} \cdot d\vec{q} = 2\pi J_i$$

γ_i is an irreducible path around a torus

$$\Delta_i \vec{\theta}_j = \frac{\partial}{\partial \vec{J}} \Delta_i S = 2\pi \frac{\partial}{\partial \vec{J}} J_i = 2\pi \delta_{ij}$$

<u>KAM Theorem</u>: Suppose in action-angle variables we perturb an integrable system

$$H(J, \theta) = H_0(\vec{J}) + \varepsilon H_1(\vec{J}, \theta)$$

What happens for small ε? Expand generating function in powers of ε

$$S = S_0 + \varepsilon S_1 + \dots$$

Fourier comp of H_1

$$S_1 = i \sum_{\vec{m}} \frac{H_{1,m}(J')}{\vec{m} \cdot \vec{\omega}_0} e^{i \vec{m} \cdot \theta}$$

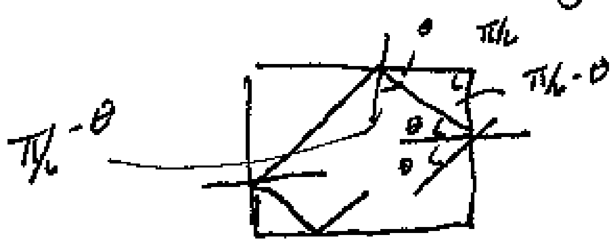
$\vec{m}$ integer

$\vec{m} \cdot \vec{\omega}_0$ can be very small

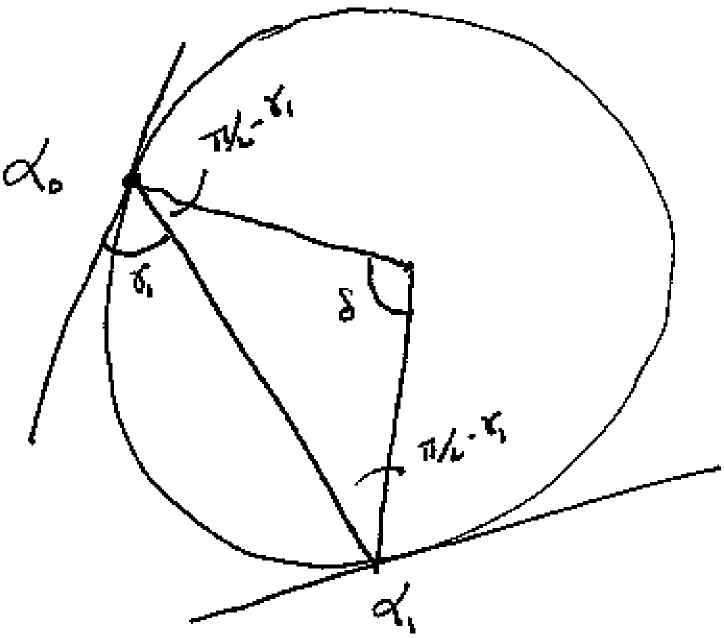
For small ε "most" of the tori survive but the "resonant tori" (for which $\vec{m}\cdot\omega_0 = 0$) for some $\vec{m}$ are destroyed. The measure of destroyed tori $\to 0$ as $\varepsilon \to 0$

Pseudo-integrable systems: Motion is not on tori but on more complex shapes - spheres with handles - Examples Motion of a particle in a polygon with angles that are rational multiples of ~~integrable billiard~~ π. ~~...~~ Richens + Berry Physica D 2, 495 (81)

Multiple handles mean actions can't be defined as above ~~so~~ system is not integrable. But not ergodic either since any trajectory can only have a fixed number of directions



Billiard Flow in a circle



$r=1$

$\delta = 2\gamma_1$

arc length $= 1\,\delta = 2\gamma_1$

$$\alpha_1 = \alpha_0 + \delta = \alpha_0 + 2\gamma_1$$

$$\alpha_n = \alpha_0 + 2n\gamma_1 \quad \text{after } n \text{ bounces}$$

Two trajectories $\delta_1' = \delta_1 + \varepsilon$, $\alpha_0' = \alpha_0$

$$\alpha_n' = \alpha_0 + 2n\gamma_1' = \alpha_0 + 2n\gamma_1 + 2n\varepsilon$$

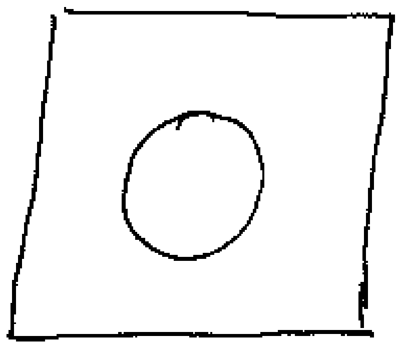
$$\gamma_1' = \gamma_1 + \varepsilon$$

$\alpha_n' - \alpha_n = 2n\varepsilon$ linear growth of trajectory separation.

There are nearby orbits that don't separate at all, α_0, $\alpha_0 + \varepsilon_0 = \alpha_0'$ $\delta' = \delta$

If δ is irrational the endpoints will cover the circle uniformly. (Weyl's theorem)

Sinai Billiard



Circular scatterers in a box

Trajectories are unstable

Exponentially separate!

Systems with exponential separation of ~~nearby~~ infinitesimally close trajectories are called chaotic.

Examples: Sinai Billiard
Bunimovich Stadium
Hard disk & sphere Lorentz gas
Gas of hard spheres.

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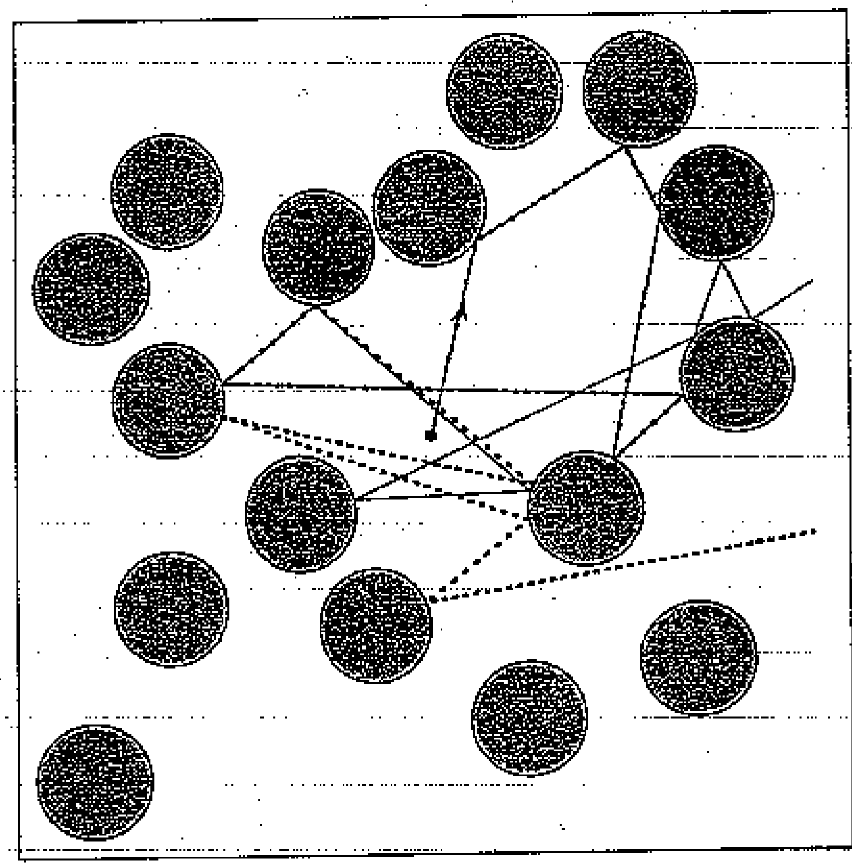
In Classical Mechanics, there are more advanced notions that we will need, to some extent.



SEPARATION OF TRAJECTORIES
IN A RANDOM, HARD DISK
LORENTZ GAS

FROM P. GASPARD

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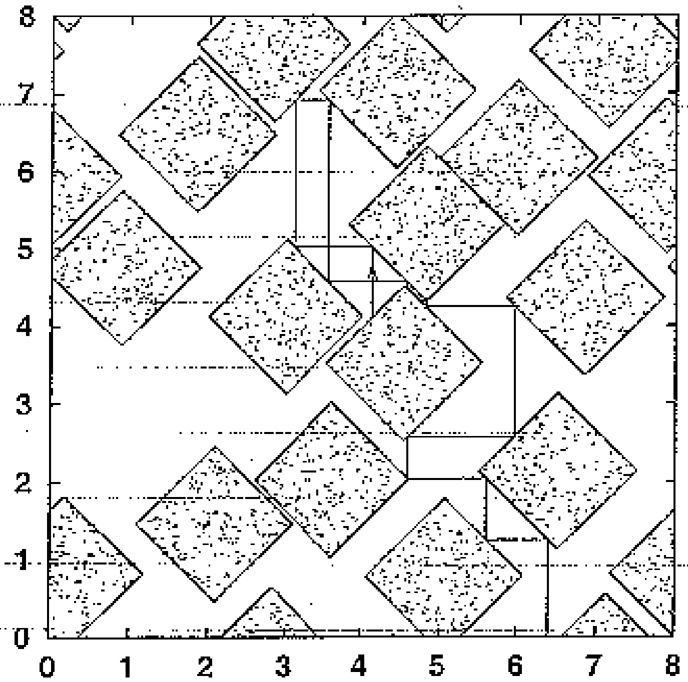


FIG. 1. The fixed orientation wind-tree model. There are periodic boundary conditions, so in the notation of section IIC this is FP8.

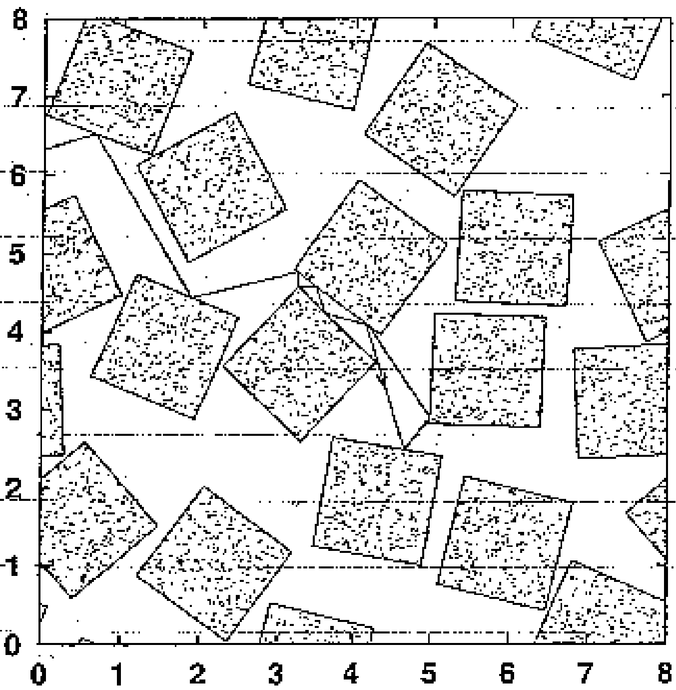
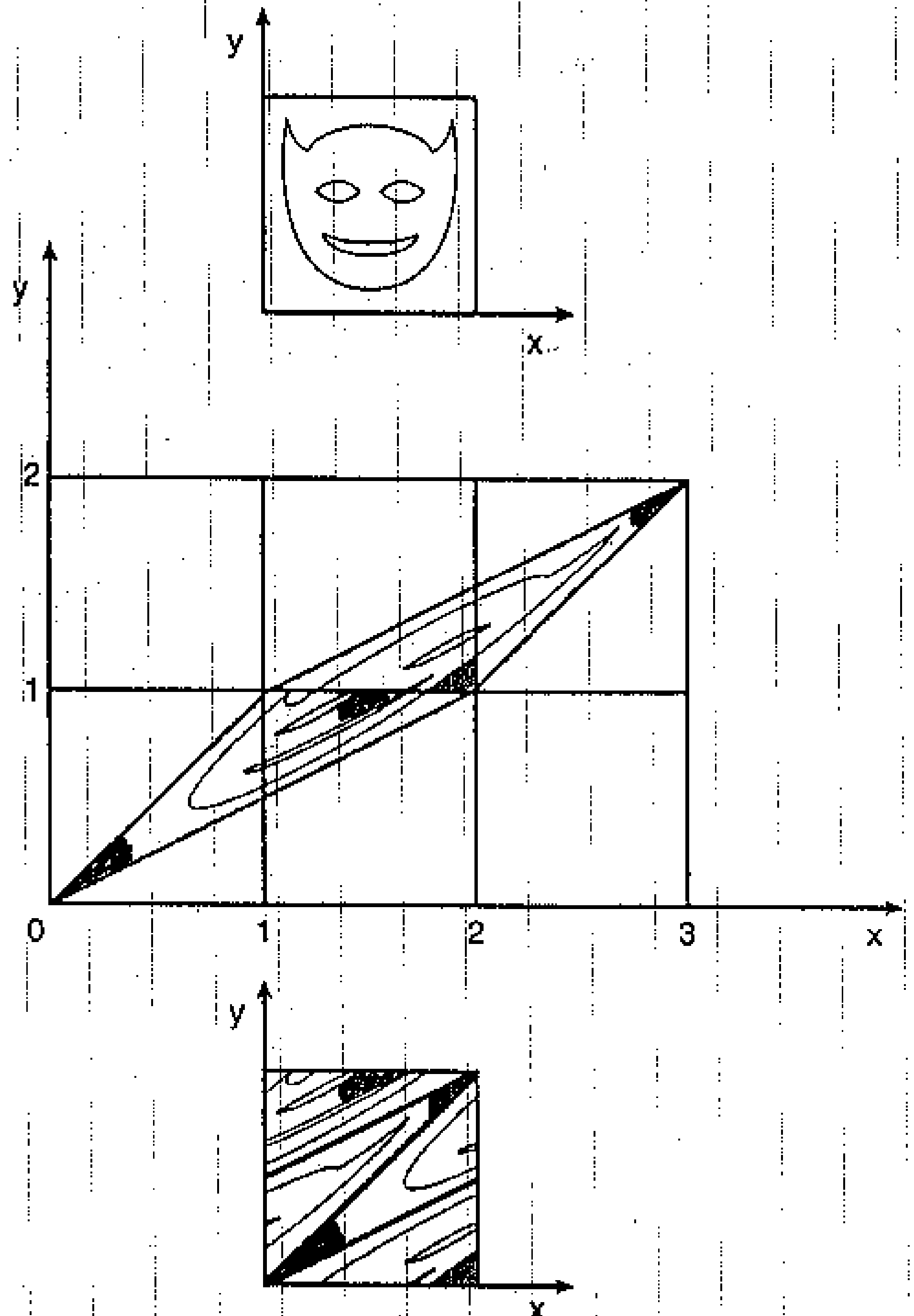


FIG. 2. The randomly oriented wind-tree model, with notation RP8.

TRAJECTORIES IN ~~A~~ RANDOM WIND-TREE MODELS

E.G.D.

FROM C. DETTMANN - V COHEN



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(Mod 1)

Arnold Cat Map

CHAOTIC MAP

Consider phase space Γ and consider constant energy surface in this phase space with an invariant measure $\mu(A) = \mu(\Phi_t' A)$

Φ_t represents the dynamics over some time t

Ergodic System: Consider any set A of positive measure $\mu(A) > 0$, let $\mu(E)$ be measure of const energy surface

Consider a trajectory let T_A / T be the fraction of time, T, that system spends in A

Def: System is ergodic if

$$\lim_{T \to \infty} \frac{T_A}{T} = \frac{\mu(A)}{\mu(E)}$$

Example: End points of circle map cover the circumference uniformly

Mixing

$$\lim_{t \to \infty} \frac{\mu(A_t \cap B)}{\mu(B)} = \frac{\mu(A)}{\mu(E)}$$

Weak Mixing

$$\lim_{T\to\infty} \frac{1}{T} \int_0^t d\tau \frac{\mu(A_\tau \cap B)}{\mu(B)} = \frac{\mu(A)}{\mu(E)}$$

Typically pseudo-integrable systems are weakly mixing - see Zaslavsky - Pseudo chaos

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Classical Statistical Mechanics on phase space Γ, $\rho(\Gamma, t) =$ ensemble distribution function can be normalized to 1

$$\frac{\partial \rho}{\partial t} + \nabla_\Gamma \cdot (\vec{V}_\Gamma \rho) = 0 \quad \text{conservation law}$$

$$\frac{\partial \rho}{\partial t} + V_\Gamma \cdot \nabla_\Gamma \rho + \rho (\vec{\nabla}_\Gamma \cdot \vec{V}_\Gamma) = 0$$

But $\vec{\nabla}_\Gamma \cdot \vec{V}_\Gamma = 0$

$$\frac{\partial \rho}{\partial t} + V_\Gamma \cdot \nabla_\Gamma \rho = 0 = \frac{d\rho}{dt} = 0 \quad \text{Liouville}$$

Conservation ρ volume in phase space

$$\frac{d\rho}{dt} = 0$$

$$\rho(\Gamma, 0) = \chi_A(\Gamma) = \begin{cases} 1 & \Gamma \in A \\ 0 & \Gamma \notin A \end{cases}$$

$$\mu(A) = \int d\Gamma \, \rho(\Gamma, 0)$$

$$\frac{d\rho(\Gamma, t)}{dt} = 0$$

$$\rho(\Gamma, t) = \chi(A_t)$$

$$\mu(A_t) = \int d\Gamma \, \rho(\Gamma, t)$$

$$\frac{d}{dt} \mu(A_t) = \frac{d}{dt} \int d\Gamma \, \rho(\Gamma, t) = \int d\Gamma \frac{d}{dt} \rho(\Gamma, t)$$

$$\int \frac{\partial \rho}{\partial t} d\Gamma = \int -\left(\dot{p}_i \frac{\partial}{\partial p_i} + \dot{\vec{r}}_i \cdot \frac{\partial}{\partial r_i}\right) \rho$$

$$= \int d\Gamma \, \rho \left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{r}_i}{\partial r_i} \right) = 0$$

Poincáre Recurrence Thm.

Almost all initial states in bounded mechanical system with fixed, finite energy, are recurrent.

~~GIVE PROOF - CLASSICAL~~

~~QUANTUM VERSION - SKETCH PROOF~~

SEE BOOKS ON ERGODIC THEORY

HALMOS, WALTERS, ...

Classical Lorentz Gases

Scatterers + moving particles

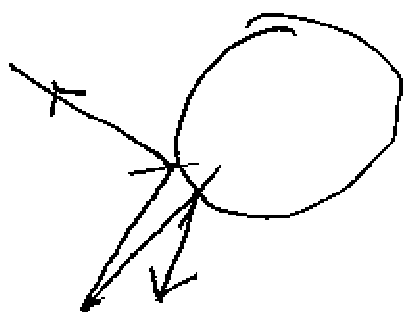
Hard disk or hard spheres.

(1) If placed on lattice sites, with no infinite horizon, then

a) Particle diffuses normally

$$\langle (x(t) - x(0))^2 \rangle = 2Dt$$

b) System is chaotic due to defocusing of the scattering process



$$\boxed{\lambda = \frac{v}{t}\int_0^t \frac{d\tau}{\rho(\tau)} \to v\left\langle \frac{1}{\rho} \right\rangle}$$

(2) If scatterers are placed at random & no traps form

a) chaotic & diffusive

(3) If there are infinite horizons can have ballistic motion over long times

If traps form

Green-Kubo Formulae

Consider diffusion - normal

GIVE DERIVATION?

$$\langle (x(t)-x(0))^2 \rangle = 2Dt$$

$$= \int_0^t d\tau_1 \int_0^t d\tau_2 \, \langle v(t_1)\, v(t_2) \rangle = 2Dt$$

$$= 2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, \langle v(t_1)\, v(t_2) \rangle = 2 \int_0^t dt_2 \int_{t_2}^t dt_1 \, \langle v(t_1)\, v(t_2) \rangle$$

$$2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, \langle v(t_1 - t_2)\, v(0) \rangle \quad u = t_1 - t_2 \quad = 2 \int_0^t dt_2 \int_{t_2}^t dt_1 \, \langle v(t_1 - t_2) \, v(0) \rangle$$

$$u = t_1 - t_2$$

$$= 2 \int_0^t dt$$

$$= 2 \int_0^t dt_2 \int_0^{t-t_2} du \, \langle v(u)\, v(0) \rangle$$

$$2Dt = 2\int_0^t dt_1 \int_0^{t_1} dt_2 \; \langle v(t_1)\, v(t_2)\rangle$$

$$= 2\int_0^t dt_1 \int_0^{t_1} dt_2 \; \langle v(t_1 - t_2)\, v(0)\rangle$$

$$= 2\int_0^t dt_1 \int_0^{t_1} du \, \langle v(u)\, v(0)\rangle$$

$$= 2\int_0^t du \, \langle v(u)\, v(0)\rangle \, (t-u) = 2Dt$$

$$D = \int_0^t du \left(1 - \frac{u}{t}\right) \langle v(0)\, v(u)\rangle$$

Mixing $\Rightarrow$ $\langle v(0)\, v(u)\rangle \longrightarrow 0$ as $\cancel{t}\, u \to \infty$

If $\int_0^t du \; u \, \langle v(0)\, v(u)\rangle \sim t^{\alpha}$ $\;\cancel{\alpha} < 1$

Then

$$D = \int_0^\infty du \, \langle v(0)\, v(u)\rangle$$ G-K Formula

Einstein !! Father of Quantum Chaos:

"Zum Quantumsatz von Sommerfeld + Epstein
Verhandlungen der Deutschen Physikalischen
Gesellschaft 19, 82 (1917)

The Bohr-Sommerfeld Quantization Condition

$$\oint p\,dq = nh$$

requires motion on a torus - i.e. that the classical system be integrabe - describable by action-angle variables. This is not true for pseudo-integrable systems (need more complex structures) and chaotic systems (no tori, etc.)

Therefore the Bohr-Sommerfeld condition is not generalizable, must have limited applicability.

Quantum Systems - Non rel, Schrodinger (13)

$$i\hbar |\dot{\alpha}\rangle = H|\alpha\rangle$$

$$|\alpha_t\rangle = e^{-\frac{iHt}{\hbar}} |\alpha_0\rangle$$

Consider bounded systems, discrete energies:

Quantum Poincare Recurrence Thm.

MENTION EARLIER! W/ CLASSICAL VERSION?

Let $\psi(\vec{r},t)$ be such that

$$\psi(\vec{r},t) = \sum_n a_n \phi_n(\vec{r}) e^{-\frac{iE_n t}{\hbar}}$$ converges uniformly

in t. Then $\psi(\vec{r},t)$ is an almost periodic fn.

Density Matrix $\underset{\sim}{\rho}$, satisfies

$$\frac{\partial \underset{\sim}{\rho}}{\partial t} = -\frac{i}{\hbar}\left(H\underset{\sim}{\rho} - \underset{\sim}{\rho}H\right)$$

$$\frac{\partial \rho}{\partial t} + \frac{i}{\hbar}[H, \underset{\sim}{\rho}] = 0$$

Pure state $\underset{\sim}{\rho}^2 = \underset{\sim}{\rho}$ + $\underset{\sim}{\rho}$ can be expressed as

$$\underset{\sim}{\rho} = |\alpha_t\rangle\langle\alpha_t|$$

Otherwise

$$\rho = \sum_i w_i \, |\alpha_i, t\rangle\langle\alpha_i, t|$$

$$\langle A\rangle_t = \sum_i w_i \langle\alpha_i, t | A | \alpha_i, t\rangle$$

$$= \mathrm{Tr}\,\rho A = \sum_i w_i \langle\alpha_i, t | A | \alpha_i, t\rangle$$

$$\mathrm{Tr}\,\rho \mathbb{1} = 1 \Rightarrow \sum w_i = 1$$

Wigner Function

$$W(\vec{R}, \vec{P}, t) = \frac{1}{(\pi\hbar)^D} \int d\vec{\eta} \, \langle\vec{R}+\vec{\eta}|\rho|\vec{R}-\vec{\eta}\rangle \, e^{-\frac{2i\vec{\eta}\cdot\vec{P}}{\hbar}}$$

Consider

$$\int d\vec{P} \, W(\vec{R}, \vec{P}, t) = \frac{1}{(\pi\hbar)^D} \int d\eta \, \langle R+\eta|\rho|R-\eta\rangle \, (2\pi)^D \delta\left(\frac{2}{\hbar}\vec{\eta}\right)$$

$$= \left(\frac{2}{\hbar}\right)^D \left(\frac{\hbar}{2}\right)^D \langle R|\rho|R\rangle$$

$$= \langle\vec{R}|\rho|\vec{R}\rangle$$

$$\int d\vec{R}\, W(\vec{R},\vec{P},t) = \frac{1}{(\pi\hbar)^D}\int d\vec{R}\int d\vec{\eta}\int\frac{d\vec{P}_1}{(2\pi\hbar)^D}\int\frac{dP_2}{(2\pi\hbar)^D}$$

$$e^{-\frac{2iP\cdot\vec{\eta}}{\hbar}}\, e^{\frac{iP_1\cdot(R+\eta)}{\hbar}}\langle P_1|\rho|P_2\rangle e^{-\frac{iP_2\cdot(R-\eta)}{\hbar}}$$

$$= \frac{1}{(\pi\hbar)^D}\cancel{(2\pi\hbar)^D}\int d\vec{\eta}\int\frac{d\vec{P}_1}{(2\pi\hbar)^D}\int\frac{dP_2}{\cancel{(2\pi\hbar)^D}}$$

$$\cancel{(2\pi\hbar)^D}\,\delta(\vec{P}_1-\vec{P}_2)\, e^{\frac{i(P_1+P_2)\eta}{\hbar}}\, e^{-\frac{2\pi i\,\vec{\eta}\cdot\vec{P}}{\hbar}}$$

$$= \frac{1}{(\pi\hbar)^D}(2\pi\hbar)^D\left(\frac{\hbar}{2}\right)^D\langle P|\rho|P\rangle$$

$$= \langle\vec{P}|\rho|\vec{P}\rangle$$

For a pure state

$$W(R,P,t) = \frac{1}{(\pi\hbar)^3}\int d\vec{\eta}\, e^{-\frac{2\pi i\,\vec{P}\cdot\vec{\eta}}{\hbar}}\,\psi^*(\vec{R}-\vec{\eta})\,\psi(\vec{R}+\vec{\eta})$$

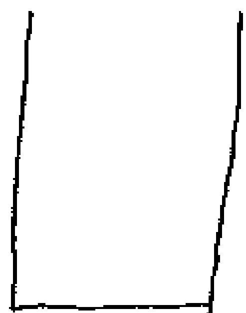
Not always positive

Sometimes its useful to define a "fuzzy Wigner Function" called a Husimi Function by

$$H_u(\vec{r},\vec{p},t) = \frac{1}{(\pi\hbar)^D}\int d\vec{r}'\int d\vec{p}'\, W(\vec{r}',\vec{p}',t)\left[e^{-\frac{(\vec{r}-\vec{r}')^2}{2a^2}-\frac{(p-p')^2a^2}{2\hbar^2}}\right]$$

Example: Particle in a 1-d box

$\theta_-(|x|) = 1 \quad |x| < \frac{L}{2}$



$-L/2 \qquad 0 \qquad L/2$

$\theta_-(|x|)$

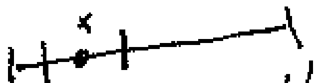
$x+\eta < \frac{L}{2}$, $\quad \eta < \frac{L}{2} - x$

$$\psi(x) = A\cos k_n x\, \theta_-(|x|)$$

$x-\eta \geq -\frac{L}{2}$, $\quad \eta < x + \frac{L}{2}$

$$k_n = \frac{(2n+1)\pi}{L}$$

$$P_n = \hbar k_n$$



x

$-L/2 \qquad L/2$

$$W(x,p) = \frac{|A|^2}{\pi\hbar}\int_{-L/2}^{L/2} d\eta\, e^{-\frac{2i\eta p}{\hbar}} \cos k_n(x+\eta)\cos k_n(x-\eta)$$

$$-\frac{L}{2} < x \pm \eta < \frac{L}{2}$$

Would like to compare classical mechanics

- takes place in Γ-space

With quantum mechanics

- takes place in $\underline{R}$-space or $\underline{P}$-space

Construct functions to interpolate between them + can be interpreted as a probability

Wigner Function - oscillates too much as $\hbar \to 0$ + has negative regions

$\downarrow$

Husimi Function - Gaussian smoothing of Wigner function

Coherent states - minimum uncertainty states

Gaussian wave functions

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$$W_n(x,p) = \frac{|A|^2}{\pi\hbar}\int^{L} d\eta \;\; \Theta_-(|x-\eta|)\,\Theta_-(|x+\eta|)$$

$$\cos(\eta+x)k_n \; \cos(x-\eta)k_n \; e^{-\frac{2i\eta p}{\hbar}}$$

$$= \frac{|A|^2}{2\hbar\pi}\int^{1} d\eta \;\; \Theta_-(|x-\eta|)\,\Theta_-(|x+\eta|)$$

$$(\cos 2k_n x + \cos 2k_n \eta)\, e^{-\frac{2i\eta p}{\hbar}}$$

$$W_n(x=0,p) = \frac{|A|^2}{2\hbar\pi}\int_{-L/2}^{L/2} d\eta\; e^{-\frac{2p\eta i}{\hbar}}\left(1+\cos 2k_n\eta\right)$$

$$= \frac{|A|^2}{2\hbar\pi}\int_{-L/2}^{L/2} d\eta\; e^{\frac{2\eta p i}{\hbar}}\left(1+\frac{1}{2}\left(e^{2ik_n\eta}+e^{-2ik_n\eta}\right)\right)$$

$$= \frac{|A|^2}{2\hbar\pi}\left[\hbar\frac{\sin \frac{pL}{\hbar}}{p} + \frac{\hbar}{2}\left(\frac{\sin \frac{(p+p_n)L}{\hbar}}{p+p_n}\right.\right.$$

$$\left.\left. + \frac{\sin\frac{(p-p_n)L}{\hbar}}{p-p_n}\right)\right]$$

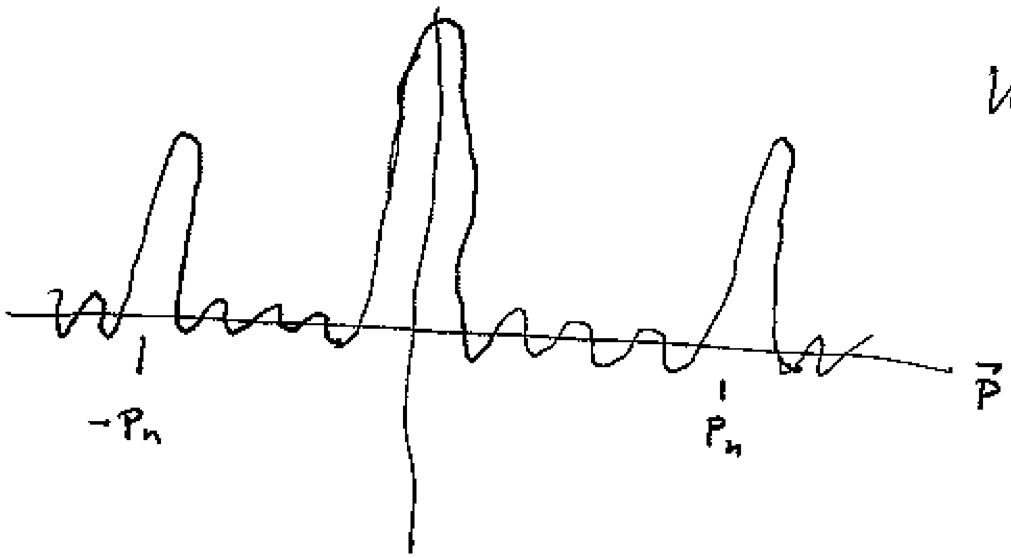
oscillates as a function of p.

$\cos^2 k\eta = \cos 2k\eta + x$

$\cos 2k\eta = \cos^2 k\eta - \sin^2 k\eta$

$= 2\cos^2 k\eta - 1$

$\cos^2 k\eta = \frac{1}{2}(1 + \cos 2k\eta)$



We can choose ~~Δ~~ in the Husimi function to get a positive H & not loose ~~too~~ much information about the quantum state. The overall positive nature of H depends on tails in the Gaussian.

Nonnenmacher + Voros ~~and~~ chao-dyn 9711016[2]
de Aguiar + Ozorio de Almeida J Phys A $\underline{23}$, L1025 (1990)

GREEN-KUBO IN QM

Linear Response

$$\frac{\partial \rho}{\partial t} - \frac{i}{\hbar}[\rho, H] = 0$$

$$H = H_0 + \varepsilon H_{ext}$$

$$\rho = \rho_0 + \varepsilon \rho_1 + \cdots$$

ε^0: $$\frac{\partial \rho_0}{\partial t} - \frac{i}{\hbar}[\rho_0, H_0] = 0$$

ε^1 $$\frac{\partial \rho_1}{\partial t} - \frac{i}{\hbar}[\rho_0, H_1] - \frac{i}{\hbar}[\rho_1, H_0] = 0$$

$\vdots$

$$\rho_0 = \rho_0^{(eq)}$$

$$\frac{\partial \rho_1}{\partial t} - \frac{i}{\hbar}\rho_0 H_1 + \frac{i}{\hbar} H_1 \rho_0 - \frac{i}{\hbar}(\rho_1 H_0 - H_0 \rho_1) = 0$$

$$\frac{\partial \rho_1}{\partial t} - \frac{i}{\hbar}(\rho_1 H_0 - H_0 \rho_1) = \frac{i}{\hbar}[\rho_0, H_1]$$

~~Consider~~ $\cancel{e^{-\frac{i}{\hbar}H_0 t}\rho_1 e^{+\frac{i}{\hbar}t H_0} = \tilde{\rho}_1}$

$$\cancel{\frac{\partial \tilde{\rho}_1}{\partial t} = \frac{-i}{\hbar}[H_0, \tilde{\rho}_1] + e^{\frac{i}{\hbar}H_0}\left[\left(+\frac{i}{\hbar}\rho_1 H_0 - \frac{i}{\hbar}H_0\rho_1\right) + \frac{i}{\hbar}(\rho_0, H_1)\right] e^{\frac{-i}{\hbar}t}}$$

$$\frac{\partial \rho_1}{\partial t} = \frac{i}{\hbar}[\rho_0, H_1] + \frac{i}{\hbar}[\rho_1 H_0 - H_0\rho_1]$$

$$\tilde{\rho}_1 = e^{\frac{i}{\hbar}tH_0}\rho_1 e^{-\frac{i}{\hbar}tH_0}$$

Cancel

$$\frac{\partial \tilde{\rho}_1}{\partial t} = \frac{i}{\hbar}[H_0, \tilde{\rho}_1] + \frac{i}{\hbar}e^{\frac{i}{\hbar}tH_0}\left[\frac{i}{\hbar}[\rho_0, H_1] + \frac{i}{\hbar}[\tilde{\rho}_1, H_0]\right]e^{-\frac{i}{\hbar}t}$$

$$\frac{\partial \tilde{\rho}_1}{\partial t} = \frac{i}{\hbar}e^{\frac{i}{\hbar}tH_0}[\rho_0, H_1]e^{-\frac{i}{\hbar}tH_0}$$

$$\tilde{\rho}_1(t) = \underset{0}{\underset{\shortparallel}{\tilde{\rho}_1(-\infty)}} + \frac{i}{\hbar}\int_{-\infty}^{t} d\tau\, e^{\frac{i}{\hbar}\tau H_0}[\rho_0, H_1]e^{-\frac{i}{\hbar}\tau H_0}$$

$$\boxed{\begin{gathered}\rho_1(t) = \frac{i}{\hbar}\int_{-\infty}^{t} e^{-\frac{i}{\hbar}(t-\tau)H_0}[\rho_0, H_1]e^{+\frac{i}{\hbar}(t-\tau)H_0} \\ \rho_0 = \frac{1}{Z}e^{-\beta H_0}\end{gathered}}$$

$$F(\beta) = [\rho_0, H_1] = \frac{1}{Z}\left[e^{-\beta H_0}H_1 - H_1 e^{-\beta H_0}\right]$$

$$\frac{dF(\beta)}{d\beta} = \frac{e^{-\beta H_0}}{Z}(-H_0 H_1) + \frac{1}{Z}H_1 H_0 e^{-\beta H_0}$$

$$\langle \vec{J} \rangle = \overset{0}{\overset{\shortparallel}{\langle \rho_0 \vec{J} \rangle}} + \varepsilon \langle \rho_1 \vec{J} \rangle + \cdots$$

$$= \varepsilon \frac{i}{\hbar} \int_{-\infty}^{t} \mathrm{Tr}\; J e^{-\frac{i}{\hbar}(t-\tau)} [\rho_0, H_1] e^{+\frac{i}{\hbar}(t-\tau) H_0}$$

$$= \varepsilon \frac{i}{\hbar} \int_{-\infty}^{t} \mathrm{Tr}\; e^{\frac{i}{\hbar}(t-\tau) H_0} \vec{J} e^{-\frac{i}{\hbar}(t-\tau) H_0} [\rho_0, H_1]$$

$$= \varepsilon \frac{i}{\hbar} \int_{-\infty}^{t} d\tau\; \mathrm{Tr}\; J(-\tau) [\rho_0, H_1]$$

Kubo transform

$$[\rho_0, H_1] = \frac{1}{Z} [e^{-\beta H_0}, H_1]$$

$$[e^{-\beta H_0}, H_1] = e^{-\beta H_0} \int_0^\beta d\lambda\, e^{\lambda H_0} [H_1, H_0] e^{-\lambda H_0}$$

$$\langle J \rangle = \varepsilon \frac{i}{\hbar} \int_{-\infty}^{t} d\tau \int_0^\beta d\lambda\; \mathrm{Tr} \{ J(-\tau) e^{-\beta H_0} e^{\lambda H_0} [H_1, H_0] e^{-\lambda H_0}$$

$$\langle J \rangle = \varepsilon \int_{-\infty}^{t} d\tau \int_{0}^{\beta} d\lambda \, \mathrm{Tr}\, J(-\tau)\, e^{-\beta H_0} e^{\lambda H_0} \dot{H}_1 e^{-\lambda H_0}$$

$$\approx \int_{-\infty}^{\infty} \langle J(-\tau) J(0) \rangle_{K.T.} \, d\tau$$

LOSCHMIDT ECHO or FIDELITY

Asher Peres - Try to find some way to see chaos in a quantum system

Let $|0\rangle$ be some initial quantum state

Consider two Hamiltonians, H_0 and $H_0 + \varepsilon H_1$ where εH_1 is a small perturbation.

Propagate $|0\rangle$ with $H_0 + \varepsilon H_1$ over a time t + then back to $t=0$ with H_0

$$|0'\rangle = e^{\frac{i}{\hbar} H_0 t} e^{-\frac{i}{\hbar}(H_0 + \epsilon H_1) t} |0\rangle$$

Compute overlap with $|0\rangle$

$$\left| \langle 0 | e^{\frac{i}{\hbar} H_0 t} e^{-\frac{i}{\hbar}(H_0 + \epsilon H_1) t} |0\rangle \right|^2 = f(t)$$

For a CCS one might expect

$$f(t) \sim e^{-\lambda t}$$ where λ is a Lyapunov exponent.

Situation is more complicated!

More later: {GORIN, PROSEN, SELIGMAN, Žnidarič PHYS REPT. **435**, 33 (2006).

Spectral Statistics of "Chaotic" & "Non Chaotic" Systems.

Random Matrix Theory RMT

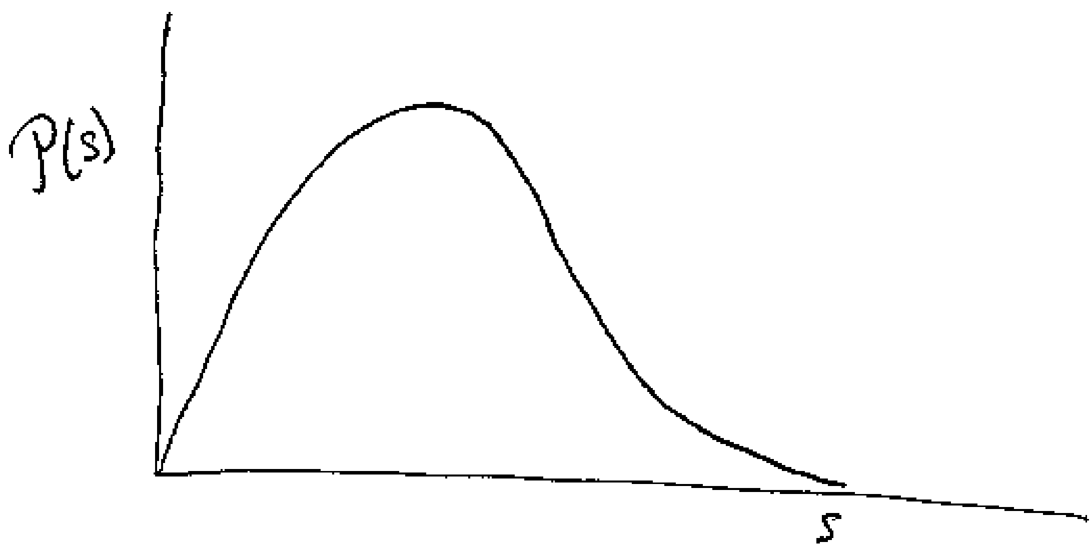
Origins in Nuclear Physics

Wigner - energy levels of large nuclei - U

Nearest neighbor spacing distribution

normalised by average spacing, D

$$s = \frac{\Delta E}{D} \qquad \langle s \rangle = 1$$



$n + U = [U{+}n]$ Resonance state is a compound nucleus "energy shared equally among all the nucleons"

quasi-equilibrium state

Wigner method

Find the typical nn spacing over an ensemble of Hamiltonians with same symmetry property. Result is generic - does not depend on particular nucleus

~~Parametric Dependence~~
~~Level Crossings~~

Energy level spacing of classically chaotic systems follow RMT. (B-G-S Conjecture: More later)

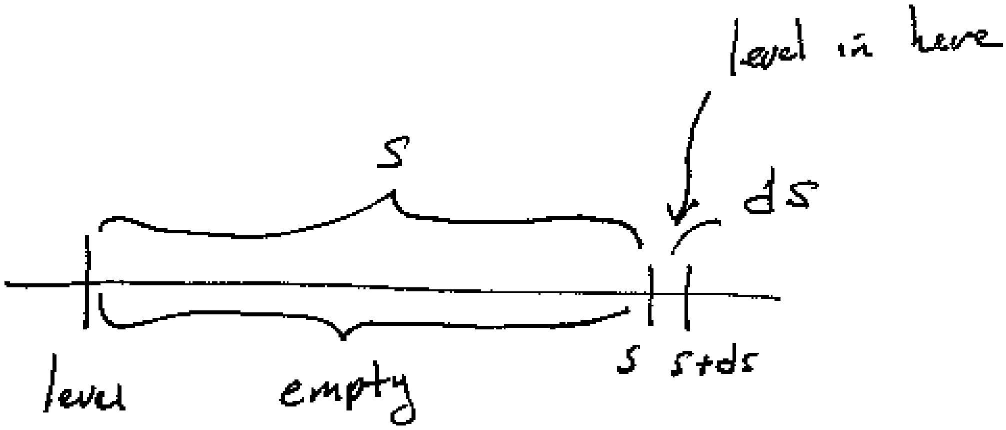
⇒ G. CASATI + F. VALZ-GRIS - Lettere al Nuovo Cimento 28, 279 (1980)

How does RMT WORK?

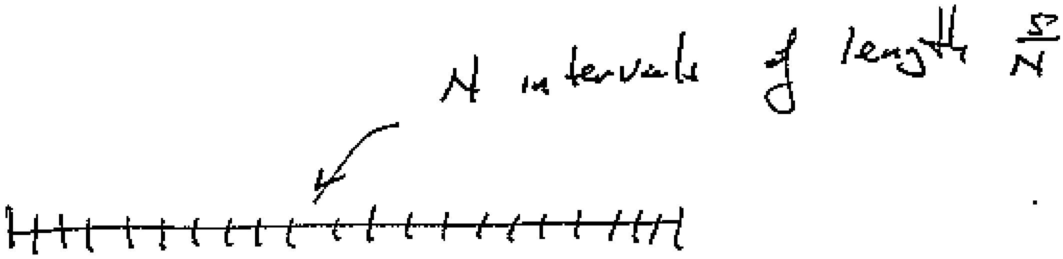
Note $P(S=0)=0$ is called "level repulsion" and is a sign of correlation between lev

Suppose no correlations, then what is $P(s)$

Compute probability that given a level, ~~there~~ that the next one is ~~a another~~ between $S + S + ds$ from it

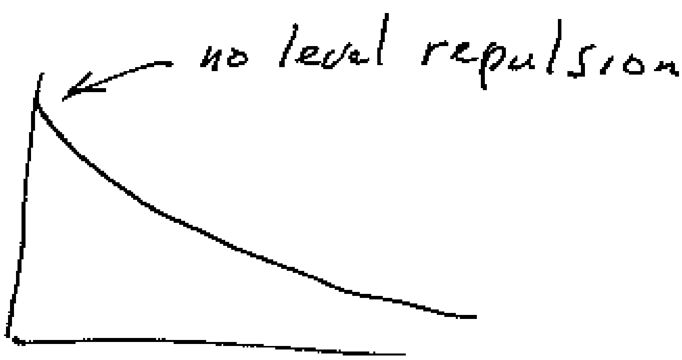


levels are uncorrelated + uniformly distributed: Prob that there is a level in an ~~distance~~ interval of length ds is ds.



$$P(s)ds = \left(1 - \frac{s}{N}\right)^N ds \longrightarrow e^{-s} ds$$

Poisson distrib



NEAREST NEIGHBOR SPACING DISTRIBUTIONS FOR EIGENVALUES FOLLOW RANDOM MATRIX DISTRIBUTIONS (GOE, GUE, GSE) FOR CHAOTIC SYSTEMS + POISSON DISTRIB. FOR NON-CHAOTIC SYSTEMS.

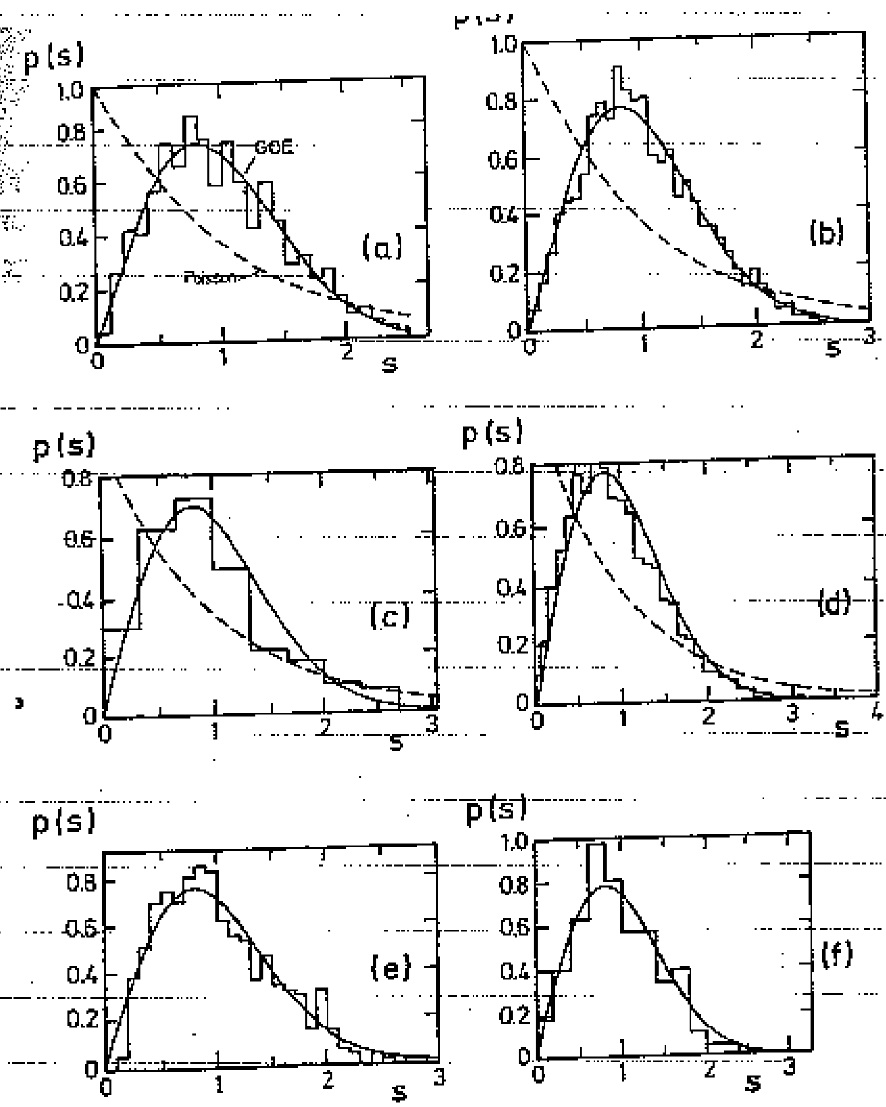


Figure 3.7. Level spacing distribution for a Sinai billiard [Boh84] (a), a hydrogen atom in a strong magnetic field [Hön89] (b), the excitation spectrum of an NO_2 molecule [Zim88] (c), the acoustic resonance spectrum of a Sinai-shaped quartz block [Oxb95] (d), the microwave spectrum of a three-dimensional chaotic cavity [Deu95] (e), and the vibration spectrum of a quarter-stadium shaped plate [Leg92] (f). In all cases a Wigner distribution is found though only in the first three cases are the spectra quantum mechanical in origin (Copyright 1984–95 by the American Physical Society).

FROM STÖCKMANN

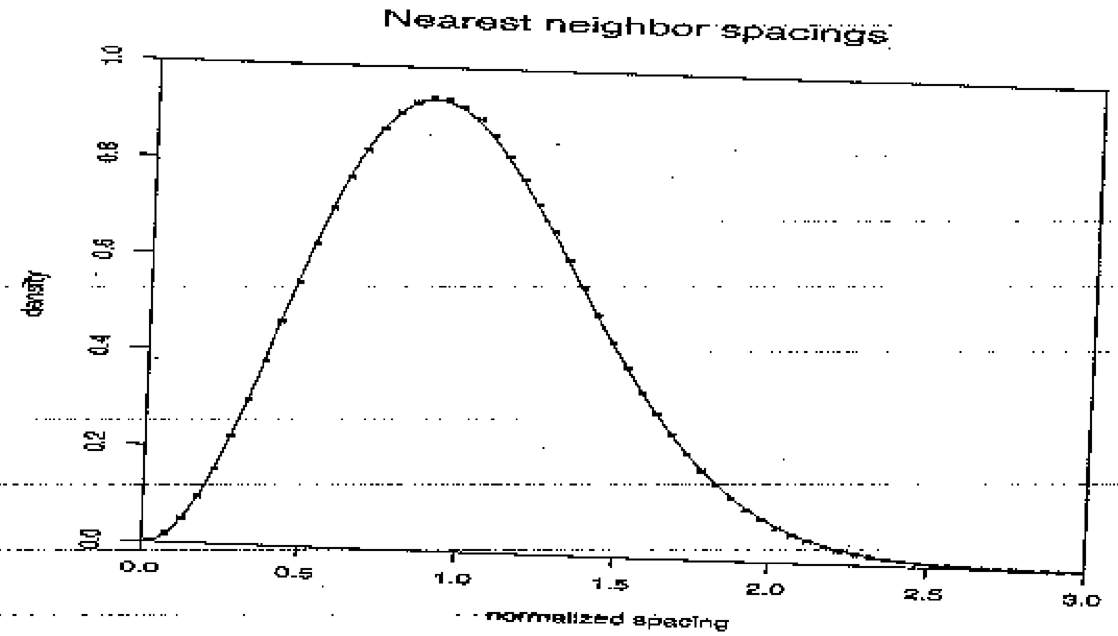


FIGURE 1. Probability density of the normalized spacings δ_n. Solid line: Gue prediction. Scatterplot: empirical data based on a billion zeros near zero # $1.3 \cdot 10^{16}$.

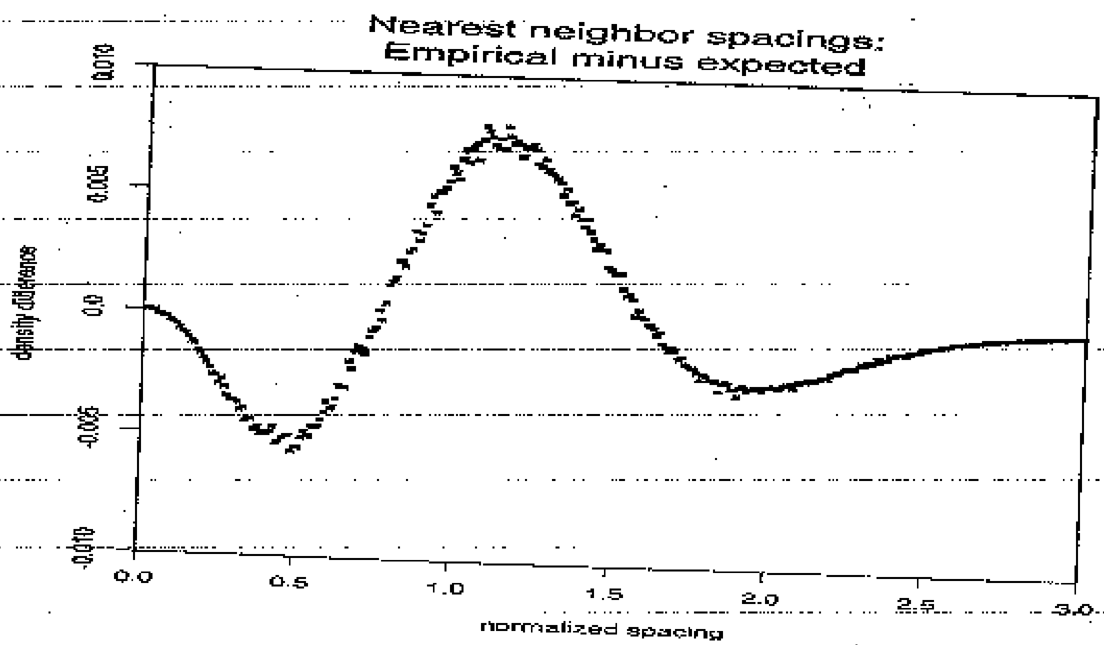


FIGURE 2. Probability density of the normalized spacings δ_n. Diffrence between empirical distribution for a billion zeros near

ZEROES OF RIEMANN ZETA FUNCTION

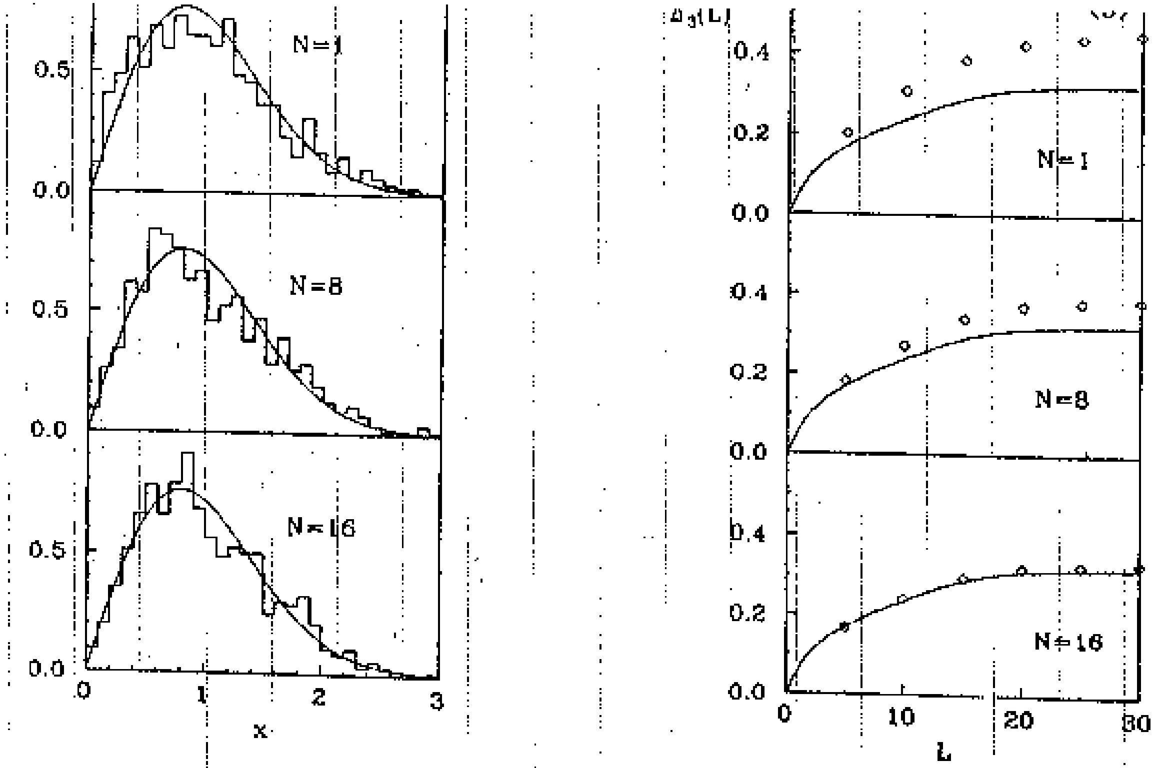


FIG. 2. (a) Nearest-level spacing distribution $P_D(x)$ for the level spectra of the square billiards $N=1$, 8, and 16. Solid lines are the predictions of the Gaussian orthogonal ensemble. (b) Dyson-Mehta statistics $\Delta_3(L)$ for the level spectra of the square billiard $N=1$, 8, and 16. Solid lines are the predictions of the Gaussian orthogonal ensemble.

RMT FOR PSEUDO - INTEGRABLE MODEL

SEE PRL of T. COHEN + T. Cheon

PRL 62, 2769 (1989).

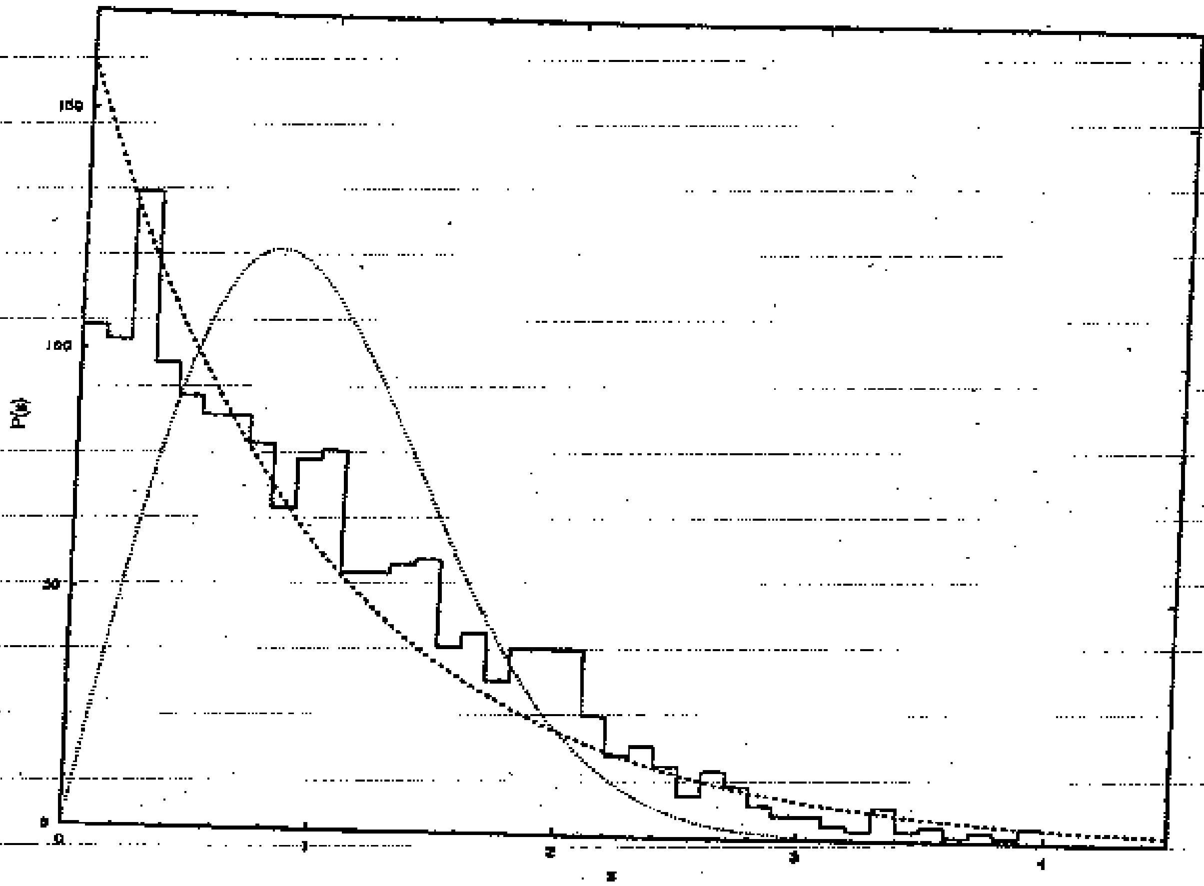


Fig. 3. The level-spacing distribution for 1700 energy levels of the triangle (2,3,∞); the dotted curve corresponds to the GOE distribution, the dashed one to Poisson.

FAILURE of RMT FOR INTERESTING REASONS. SEE PAPER OF BOGOMOLNY ETAL

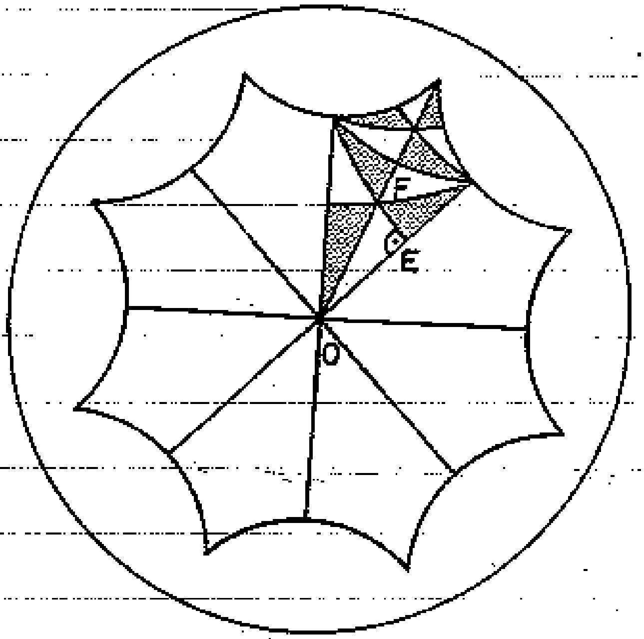


Figure 60 The regular octogon is divided into 96 congruent triangles with the inner angles $\pi/2$, $\pi/3$, and $\pi/8$; Schmit calculated the spectrum of the Laplacian with Dirichlet boundary conditions for this triangle.

SEE PAPER OF BOGOMOLNY ET AL

This is <u>not</u> what one sees in large, complex nuclei or in the spectra of CS (Classically Chaotic systems) [Berry, R. Marcus, Zaslavsky, Guarneri, Casati, McDonald, Kaufman ..

~~FIGURE FROM BROCKMAN HERE~~

GAUSSIAN RANDOM MATRIX ENSEMBLES

I. SYMMETRIES of Hamiltonians _

$\underset{\sim}{H}$ is a Hermitian Matrix. in any basis

$$H_{mn} = \langle m | H | n \rangle = H^*_{nm}$$

H may have other symmetries - rotational, time reversal invariance, etc. In a given basis H is a block-diagonal matrix, where all elements in each block correspond to a given values of a set of quantum numbers

We know from classical Mechanics that for Chaotic Systems, only the most obvious constants of the motion exist, associated with simple symmetries: Among them

① Energy - time translation
② Angular Momentum - Rotation
③ Linear Momentum - Spatial translations

Invariances

For large nuclei the high energy levels become so nearly degenerate that many quantum numbers are meaningless in trying to characterize the states. Here too one looks for "Universal symmetries" whose numbers can classify large blocks in the H-matrix

Dyson's Three Fold Way (Wigner)
J. Math Phys 3, 1189 (1962)

Hamiltonian Matrices, ie Hermitian Matrices Fall into one of 3 universality classes, depending upon their properties under time-reversal.
These are transformations of matrices that

~~Al Matrices that~~

Preserve
1) Hermiticity
2) Eigenvalues

These are
1) ~~Unitary~~ Orthogonal Transformations - Systems of Even Spin + time reversal invariant Hamiltonian
2. Symplectic ~~Ensemble~~ Transformations - odd spin + time reversal invariant
3. ~~Orthogonal~~ Unitary Ensemble - No invariance under time reversal

Each ~~These~~ ~~of~~ the classes ~~are~~ is preserved under a ~~[illegible]~~ similarity transformation from the corresponding group

$$\Omega H \Omega^{-1} = H' \quad \text{is in the same class}$$

When $\Omega =$ ~~unitary~~ ortogonal, symplectic or unitary matrix, respectively

---

Examples: Construct Time reversal operator, T

Classically

$$T\vec{x} = \vec{x}$$

$$T\vec{p} = -p$$

$$Tt = -t$$

Magnetic field changes sign

$$\nabla \times B - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{\vec{J}}{c}$$

If $\vec{J}$ + t change sign then

$$E \to E$$

$$B \to -B$$

& Maxwell's eq is invariant

Q.M.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

If $\psi(\vec{r},t)$ is a solution then so is

$\psi^*(\vec{r},-t)$ since if V is real

$$-i\hbar \frac{\partial \psi^*(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*$$

But if $t \to -t$

$$i\hbar \frac{\partial \psi^*(\vec{r},-t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^*(\vec{r},-t) + V\psi^*(\vec{r},-t)$$

Let K = complex conjugation operator

$K\psi = \psi^*(\vec{r},t)$

So if $\psi(\vec{r},t)$ is a solution then so is

$K\psi(\vec{r},-t)$

K satisfies

1) $K[a\psi_1 + b\psi_2] = a^* K\psi_1 + b^* K\psi_2$

Anti-linear Operator

$$\langle K\psi_1 | K\psi_2\rangle = \int dr\,(K\psi_1)^*(K\psi_2) = \int dr\,\psi_2^*\,\psi_1$$

$$= \langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle^*$$

anti-unitary

For particles without spin time reversal operator = K

$K^2 = \mathbb{1}$ $\quad K = K^{-1}$

$K\vec{r}K^{-1} = \vec{r}$

$K\vec{p}K^{-1} = -p$

$K\vec{J}K^{-1} = -\vec{J}$

etc.

K = comp conj in position rep

K depends on represent!

For a system with H invariant under time reversal, all nondegenerate states have real eigenfunctions & we can construct a basis in terms of real functions

Note that $[H, K] = 0$ so can have common eigenfunctions of H & K, ϕ_n,

$$K\phi_n = \phi_n^* = \pm\phi_n$$

ϕ_n is real

ϕ_n is imaginary

Anti unitary property

$$\langle K\psi_1 | K\psi_2\rangle = \langle \psi_1|\psi_2\rangle^* = \langle\psi_2|\psi_1\rangle$$

$\psi_1 = \psi$

$\psi_2 = T\psi$

$$\underbrace{\langle T\psi|\psi\rangle = \langle T\psi|T^2\psi\rangle}_{\text{Anti unitary}} \underset{T^2=-1}{=} -\langle T\psi|\psi\rangle = 0$$

Kramers degeneracy

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$T\sigma_x T^{-1} = -\sigma_x$

$T\sigma_y T^{-1} = -\sigma_y$

$T\sigma_z T^{-1} = -\sigma_z$

$T = UK$

then $U\sigma_x U^{-1} = -\sigma_x$

$U\sigma_y U^{-1} = \sigma_y$

$U\sigma_z U^{-1} = -\sigma_z$

$U = i\sigma_y$

In that case H_{nm} is a real symmetric matrix & ~~it~~ this is preserved under <u>orthogonal transformations</u> – Orthogonal ensemble

For spin ½ particles

$$T = e^{\frac{i\pi}{2}\sigma_y} K$$

Kramers degeneracy if $T^2 = -1$

$$T^2 = e^{\frac{i\pi}{2}\cdot 2\sigma_y} = e^{i\pi\sigma_y} = \mathbb{1}\cos\pi + i\sigma_y \sin\pi = \underline{-1}$$

$$H|\psi_n\rangle = E_n|\psi\rangle$$

$$T H|\psi_n\rangle = H|T\psi_n\rangle = E_n T|\psi_n\rangle$$

So $|\psi\rangle$ & $|T\psi\rangle$ are both eigenstates

Now

$$\langle\psi|T\psi\rangle = \langle T^2\psi|T\psi\rangle = -\langle\psi|T\psi\rangle = 0$$

If T is anti-unitary. $|\psi\rangle$ & $T|\psi\rangle$ are different – Kramers degeneracy

Spin ½ time reversible systems

we need $\vec{\sigma}, I \sim \vec{\tau}, I$ where $\vec{\tau} = i\vec{\sigma}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tau_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\tau_a^2 = -I$$

τ's anti commute (as do σ)

H can be written

$$H = H_0 I + \sum_{i=1}^{3} H_i \tau_i$$

H_i $i = 0, \ldots 3$ are real operators

and H commutes with $T = e^{\frac{\pi}{2}\tau_y} K$

$$= \left[\cos\tfrac{\pi}{2} I + i \sin\tfrac{\pi}{2} \sigma_y\right] K = i\sigma_y K$$

$$= \tau_y K$$

$$[T, \tau_x] = [\tau_y K, \tau_x] = \tau_y K \tau_x - \tau_x \tau_y K$$

$K\tau_x = -\tau_x$

$$[T, \tau_x] = -\tau_y \tau_x K - \tau_x \tau_y K$$
$$= \tau_x \tau_y K - \tau_x \tau_y K = 0$$

<u>Oh</u>.

This is Kramers degen.

Also one can show that it is possible to choose a basis where

$$H_{nm} = H_{0,nm}\,\mathbb{I} + \sum_{i=1}^{3} H_{i,nm}\,\tau_i$$

where ~~All~~ $H_{i,nm}$ $i=0,\dots,3$ are <u>real</u>

These are quaternion-real matrices. This property is preserved by symplectic transformations

$$H' = S H S^R$$ S^R is dual to S

$$S S^R = 1, \quad S^R = Z S^T Z^{-1} = -Z S Z$$

$$Z = \begin{bmatrix} \tau_y & & \\ & \ddots & \\ & & \ddots \end{bmatrix}$$

Z is blockdiagonal with τ_y along blocks

A symplectic matrix M satisfies

$$M^T \Omega M = \Omega$$

$$\Omega = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}$$

See Stöckmann

SUMMARY

1) Hamiltonians without time reversal symmetry are represented by an ensemble Hermitian matrices, which is invariant under unitary transformations

2. Time reversal symmetry and integer spin
ensemble H-matrices invariant under orthogonal transf.

3. Time reversal, ~~½~~ n+½ spin,
ensemble invariant under symplectic transformations

NO OTHER SYMMETRIES - MESS UP THE ENSEMBLE

~~EXPTL EXAMPLES~~ APPLICATIONS OF O + U ENSEMBLES ARE KNOWN (BILLIARDS ARE GOOD)

ENSEMBLE DISTRIBUTIONS:

We want to calculate the Prob that 2 adjace[nt] levels are spaced by a distance S, where things are normalized by average spacing

We ask for distribution of Matrix elements of H.

① Prob distribution should be unchanged if

$\underset{\sim}{H}$ is replaced by $\underset{\sim}{M}^{+}\underset{\sim}{H}\underset{\sim}{M}$

Where M is O, U, or S

② Elements of $\underset{\sim}{H}$ are independent except for Hermitian symmetries $H_{ij} = H_{ji}^{*}$

We take H_{ij} to be independent for $i \leq j$

Examples of 2x2 Matrices (Haake)

We need $P(H_{11}, H_{22}, H_{12})$

normalized to unity

Independence: $P(H_{11}, H_{22}, H_{12}) = P_{11}(H_{11})\, P_{22}(H_{22})\, P_{12}(H_{12})$

(a) Orthogonal ensemble $\Theta^T H \Theta$.

Infinitesimal $\underset{\sim}{\Theta} = \begin{bmatrix} 1 & -\theta \\ \theta & 1 \end{bmatrix}$

$$H' = \Theta^T H \Theta = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \underbrace{\begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \begin{bmatrix} 1 & -\theta \\ \theta & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \begin{bmatrix} H_{11} + \theta H_{12} & -H_{11}\theta + H_{12} \\ H_{12} + \theta H_{22} & -\theta H_{12} + H_{22} \end{bmatrix}$$

$$H_{11}' = H_{11} + 2\theta H_{12}$$

$$H_{12}' = H_{12} - \theta(H_{11} - H_{22})$$

$$H_{22}' = H_{22} - 2\theta H_{12}$$

$$P_{11}(H_{11}')\,P_{22}(H_{22}')\,P_{12}(H_{12}')$$

$$= P_{11}(H_{11}+2\theta H_{12})\,P_{22}(H_{22}-2\theta H_{12})\,P_{12}(H_{12}-\theta(H_{11}-H_{22}))$$

$$= \left[P_{11}(H_{11}) + 2\theta H_{12}\frac{dP_{11}}{dH_{11}}\right]\left[P_{22}(H_{22}) - 2\theta H_{12}\frac{dP_{22}}{dH_{22}}\right]$$

$$\left[P_{12}(H_{12}) - \theta\,(H_{11}-H_{22})\frac{dP_{12}}{dH_{12}}\right]$$

$$= P_{11}(H_{11})\,P_{22}(H_{22})\,P_{12}(H_{12})\cdot$$

$$\left[1 + \theta\left\{2H_{12}\frac{d\ln P_{11}}{dH_{11}} - 2H_{12}\frac{d\ln P_{22}}{dH_{22}} - \theta(H_{11}-H_{22})\frac{d\ln P_{12}}{dH_{12}}\right\} + O(\theta^2)\right]$$

$$\{\quad\} = 0$$

$$\frac{1}{H_{12}}\frac{d\ln P_{12}}{dH_{12}} - \frac{2}{H_{11}-H_{22}}\left(\frac{d\ln P_{11}}{dH_{11}} - \frac{d\ln P_{22}}{dH_{22}}\right) = 0$$

$$\frac{1}{H_{12}} \frac{d \ln P_{12}}{d H_{12}} = \alpha$$

$$\frac{2}{H_{11} - H_{22}} \left(\frac{d \ln P_{11}}{d H_{11}} - \frac{d \ln P_{22}}{d H_{22}} \right) = \alpha$$

$$\cancel{P_{12} e^{\alpha H_{12}}}$$

$$\frac{d \ln P_{12}}{d H_{12}} = \alpha H_{12}$$

$$\ln P_{12} = \frac{\alpha}{2} H_{12}^2$$

$$P_{12} = c_{12} e^{\frac{\alpha}{2} H_{12}^2}$$

$$\frac{d \ln P_{11}}{d H_{11}} - \frac{d \ln P_{22}}{d H_{22}} = \frac{\alpha}{2} (H_{11} - H_{22})$$

$$\frac{d \ln P_{11}}{d H_{11}} - \frac{\alpha}{2} H_{11} = \frac{d \ln P_{22}}{d H_{22}} - \frac{\alpha}{2} H_{22}$$

$$\frac{d \ln P_{11}}{d H_{11}} - \frac{\alpha}{2} H_{11} = \beta$$

$$\frac{d \ln P_{22}}{d H_{22}} - \frac{\alpha}{2} H_{22} = \beta$$

$$\ln P_{11} = \frac{\alpha}{4} H_{11}^2 + \beta H_{11} + C_{11}$$

$$\ln P_{22} = \frac{\alpha}{4} H_{22}^2 + \beta H_{22} + C_{22}$$

$$P = C e^{\frac{\alpha}{4}\left[H_{11}^2 + H_{22}^2 + 2H_{12}^2\right] + \frac{\alpha\beta}{4}(H_{11}+H_{22})}$$

$$H^2 = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

$$= \begin{bmatrix} H_{11}^2 + H_{12}^2 & x \\ x & H_{12}^2 + H_{22}^2 \end{bmatrix}$$

$$P = C e^{\frac{\alpha}{4}\mathrm{Tr}\, H^2 + \frac{\alpha\beta}{4}\mathrm{Tr}\, H}$$

Adjust energy levels so that $\mathrm{Tr}\, H = 0$

$$\boxed{P(H) = C e^{-A\,\mathrm{Tr}\, H^2}}$$

Unitary Case $\qquad P_{12}(\mathrm{Re}\,H_{12}, \mathrm{Im}\,H_{12})$

$$P(H) = P_{11}(H_{11})\, P_{22}(H_{22})\, \cancel{P_{12}(H_{12})\, P_{12}(H_{12}^*)}$$

4 variables $\quad H_{11},\ H_{22},\ \mathrm{Re}\,H_{12},\ \mathrm{Im}\,H_{12}$

Infinitesimal Unitary Matrix

$$U = \mathbb{1} - i\vec{\varepsilon}\cdot\vec{\sigma}$$

$$U^\dagger = \mathbb{1} + i\varepsilon\cdot\sigma$$

$$H' = H + i\left[(\varepsilon\cdot\sigma), H\right] + O(\varepsilon^2)$$

$$\varepsilon\cdot\sigma = \varepsilon_x\sigma_x + \varepsilon_y\sigma_y + \varepsilon_z\sigma_z$$

$$= \begin{pmatrix} 0 & \varepsilon_x \\ \varepsilon_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\varepsilon_y \\ i\varepsilon_y & 0 \end{pmatrix} + \begin{pmatrix} \varepsilon_z & 0 \\ 0 & -\varepsilon_z \end{pmatrix}$$

$$= \begin{bmatrix} \varepsilon_z & \varepsilon_x - i\varepsilon_y \\ \varepsilon_x + i\varepsilon_y & -\varepsilon_z \end{bmatrix}$$

$$\Sigma\cdot\sigma\cdot H - H\cdot\Sigma\cdot\sigma$$

$$= \begin{bmatrix} \Sigma_t & \tilde{\Sigma} \\ \tilde{\Sigma}^* & -\Sigma_t \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}$$

$$- \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \begin{bmatrix} \Sigma_t & \tilde{\Sigma} \\ \tilde{\Sigma}^* & -\Sigma_t \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_t H_{11} + \tilde{\Sigma} H_{12}^* & \Sigma_t H_{12} + \tilde{\Sigma} H_{22} \\ \tilde{\Sigma}^* H_{11} - \Sigma_t H_{12}^* & \tilde{\Sigma}^* H_{12} - \Sigma_t H_{22} \end{bmatrix}$$

$$- \begin{bmatrix} H_{11}\Sigma_t + H_{12}\tilde{\Sigma}^* & H_{11}\tilde{\Sigma} - H_{12}\Sigma_t \\ H_{12}^*\Sigma_t + H_{22}\tilde{\Sigma}^* & H_{12}^*\tilde{\Sigma} - \Sigma_t H_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{\varepsilon} H_{12}^* - H_{12}\tilde{\varepsilon}^* & 2H_{12}\varepsilon_z + \tilde{\varepsilon}(H_{22} - H_{11}) \\ \\ \\ \tilde{\varepsilon}^*(H_{11} - H_{22}) - 2H_{12}^*\varepsilon_z & \tilde{\varepsilon}^* H_{12} - H_{12}^*\tilde{\varepsilon} \end{bmatrix}$$

$$\tilde{\varepsilon} H_{12}^* - H_{12}\tilde{\varepsilon}^* = (\varepsilon_x - i\varepsilon_y)(H_{12}^r - iH_{12}^i)$$

$$- (\varepsilon_x + i\varepsilon_y)(H_{12}^r + iH_{12}^i)$$

$$= \varepsilon_x \left[H_{12}^r - iH_{12}^i - H_{12}^r - iH_{12}^i \right]$$

$$- i\varepsilon_y \left[H_{12}^r - iH_{12}^i + H_{12}^r + iH_{12}^i \right]$$

$$= \varepsilon_x(-2iH_{12}^i) - i\varepsilon_y(2H_{12}^r)$$

$$= -2i(\varepsilon_x H_{12}^i + \varepsilon_y H_{12}^r)$$

$$= -\cancel{2i}\left[\varepsilon_x \frac{(H_{12} - H_{12}^*)}{\cancel{2i}} + i\varepsilon_y \frac{(H_{12} + H_{12}^*)}{\cancel{2}}\right]$$

$$= \cancel{\varepsilon_x 2 H_{12}} H_{12}(-\varepsilon_x - i\varepsilon_y) + H_{12}^*(\varepsilon_x - i\varepsilon_y)$$

$$= -H_{12}(\varepsilon_x + i\varepsilon_y) + H_{12}^*(\varepsilon_x - i\varepsilon_y)$$

$$(\varepsilon_x + i\varepsilon_y)\left[H_{12}\left(-\frac{d\ln P_{11}}{dH_{11}} + \frac{d\ln P_{22}}{dH_{22}}\right) + (H_{11} - H_{22})\frac{\partial \ln P_{12}}{\partial H_{12}^*}\right]$$

$$+ 2\varepsilon_z H_{12}\frac{\partial \ln P_{12}}{\partial H_{12}} - \text{c.c.} = 0$$

Eventually, letting $\operatorname{Tr} H = 0$

We also get

$$P(\underset{\sim}{H}) = C e^{-A \operatorname{Tr} H^2}$$ for Unitary Ensembles.

Symplectic case is sketched by Haake

Infinitesimal Symp trans

$$S = \begin{bmatrix} 1 - \varepsilon \cdot z & \alpha \\ -\alpha & 1 + \varepsilon \cdot z \end{bmatrix}$$

all are 2×2 blocks & $\underset{\sim}{H}$ is 4×4

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

h's are 2×2 blocks.

$h_{21} = (h_{12}^*)^T$

In all cases, with $\mathrm{Tr}\, H = 0$

$$P(\underset{\sim}{H}) = C e^{-A \,\mathrm{Tr}\, H^2}$$

Now we need to get the spacing distribus

Orthogonal case

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

$$H_{11}^2 + H_{22}^2 + 2H_{11}H_{22}$$
$$- 4H_{11}H_{22} + 4H_{12}^2$$
$$= (H_{11} - H_{22})^2 + 4H_{12}^2$$

$$E_1 + E_2 = H_{11} + H_{22}$$

$$E_1 \cdot E_2 = H_{11}H_{22} - H_{12}^2$$

$$E_1 \cdot [H_{11} + H_{22} - E_1] = H_{11}H_{22} - H_{12}^2$$

$$E_1^2 - E_1(H_{11} + H_{22}) + \mathrm{Det}\, H = 0$$

$$E_1 = \frac{1}{2}\left[H_{11} + H_{22} \pm \sqrt{(H_{11} + H_{22})^2 - 4\,\mathrm{Det}\, H}\,\right]$$

$$E_\pm = \frac{1}{2}\left[(H_{11} + H_{22}) \pm \left[(H_{11} - H_{22})^2 + 4H_{12}^2\right]^{1/2}\right]$$

$$\begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix} = \Theta^T H \,\Theta$$

$$\underset{\sim}{H} = \Theta \underset{\sim}{E} \Theta^T$$

Take $\Theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$H = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$H_{11} = E_+ \cos^2\theta + E_- \sin^2\theta$$

$$H_{22} = E_+ \sin^2\theta + E_- \cos^2\theta$$

$$H_{12} = (E_+ - E_-) \sin\theta \cos\theta$$

Change variables from (H_{11}, H_{22}, H_{12}) to (E_+, E_-, θ)

$$P(H)\, dH_{11}\, dH_{22}\, dH_{12} = P(E_+, E_-, \theta)\, dE_+\, dE_-\, d\theta\, J$$

↑ Jacobian

$$
J = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ 2\sin\theta\cos\theta\,(E_+ - E_-) & -2\sin\theta\cos\theta\,(E_+ - E_-) & (E_+ - E_-)\cos 2\theta \end{bmatrix}
$$

$$
= (E_+ - E_-)\, f(\theta)
$$

$$
f(\theta) = \mathrm{Det} \begin{vmatrix} \cos^2\theta & \sin^2\theta & \frac{1}{2}\sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\frac{1}{2}\sin 2\theta \\ \sin 2\theta & -\sin 2\theta & \cos 2\theta \end{vmatrix}
$$

Integrating over θ we get

$$
P_0(E_+, E_-) = C_0 |E_+ - E_-|\, e^{-A_0(E_+^2 + E_-^2)}
$$

For Unitary ensemble we need unitary matrix to diagonalize H & has two parameters

$$
U = \begin{bmatrix} \cos\theta & -\sin\theta\, e^{-i\theta} \\ \sin\theta\, e^{+i\phi} & \cos\theta \end{bmatrix}
$$

Jacobian $\sim |E_+ - E_-|^2$

$$P_u(E_+, E_-) = C_u |E_+ - E_-|^2 e^{-A_u(E_+^2 + E_-^2)}$$

For the symplectic case, we have the Kramers degeneracy so although $H = 4\times 4$ only 2 diff eigenvalues, E_+, E_-, The number of independent matrix elements of $\underset{\sim}{H}$ is 6 $[E_+, E_-, \alpha, \underset{\sim}{\xi}]$
α is an angle

The 6 equations are enough to get

$$P_s(E_+, E_-) = C_s |E_+ - E_-|^4 e^{-A_s(E_+^2 + E_-^2)}$$

GOE, GUE, GSE!

$$P(E_+, E_-) = C_\beta |E_+ - E_-|^\beta e^{-A_\beta(E_+^2 + E_-^2)}$$

$\beta = 1$ GOE
$\beta = 2$ GUE
$\beta = 4$ GSE

Normalized Distributions

$$S = \frac{\Delta E}{D}$$

$$\langle S \rangle = \int_0^\infty ds\, S P(s) = 1$$

$$\int_0^\infty ds\, P(s) = 1$$

$$P(s) = \frac{s\pi}{2} e^{-\frac{\pi s^2}{4}} \qquad \text{GOE}$$

$$\frac{32 s^2}{\pi^2} e^{-\frac{4 s^2}{\pi}} \qquad \text{GUE}$$

$$\frac{2^{18} s^4}{3^6 \pi^3} e^{-\frac{64 s^2}{9\pi}} \qquad \text{GSE}$$

NEED ILLUSTRATIONS

EXPTL

Systems with Time Periodic Hamiltonians,

Floquet Systems (Gaston Floquet c1879)

$$H(t) = H_0 + V(t)$$

$$V(t) = \sum_n A_n \, \delta(t - n\tau) \quad \text{Kicked Systems}$$

H is invariant under $H(t) \to H(t+\tau)$

Eigenfunctions are also eigenfunctions of T_τ: $\psi_i(t+\tau) = T_\tau \, \psi_i(t) = \lambda_i \, \psi_i(t)$

Since $\psi(t)$ is normalized an an eigenfunction of $H(t)$, $|\lambda_i|^2 = 1$ and $\lambda_i = e^{-i\varphi_i}$

and

$$\psi_i(t) = e^{-i\omega_i t} \, u_i(t)$$

$$u_i(t+\tau) = u_i(t)$$

$$\varphi_i = \omega_i \tau$$

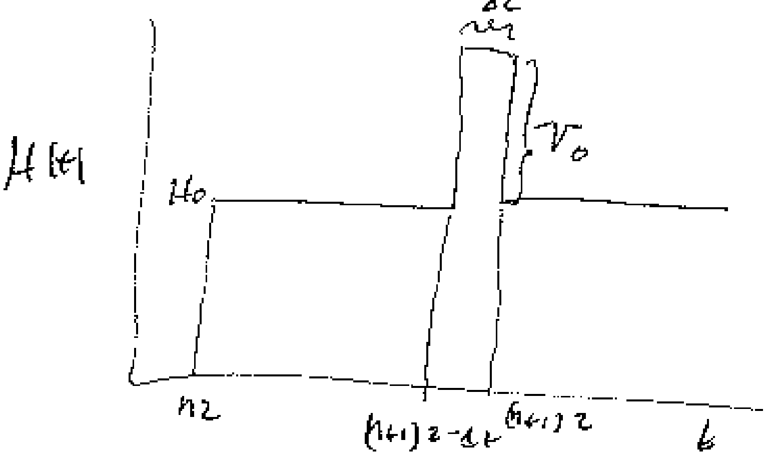
$\hbar \omega_i$ is called the quasi-energy

Suppose $H(t) = H_0 + V_0 \sum_n \delta(t - n\tau)$

We represent the δ-fn as a step function of width $\Delta\tau$ & height $\frac{1}{\Delta\tau}$

$$H(t) = H_0 \qquad n\tau < t < (n+1)\tau - \Delta\tau$$

$$H_0 + \frac{V_0}{\Delta\tau} \qquad (n+1)\tau - \Delta t < t < (n+1)\tau$$



$\cancel{\psi(t+n\tau)}$ $\cancel{\psi(t+(n+1)\tau-\Delta\tau) = e^{-iH_0}}$

$$\psi(n\tau + t) = e^{-\frac{iH_0 t}{\hbar}} \psi(n\tau) \qquad 0 < t < \tau - \Delta\tau$$

$$\psi((n+1)\tau) = e^{-\frac{i}{\hbar}\left[H_0 + \frac{V_0}{\Delta\tau}\right]\Delta\tau} e^{-\frac{iH_0}{\hbar}(\tau - \Delta\tau)} \psi(n\tau)$$

As $\Delta\tau \to 0$

$$\psi((n+1)\tau) = \underbrace{e^{-\frac{i}{\hbar} V_0} e^{-\frac{i}{\hbar} H_0 \tau}}_{\text{FLOQUET OPERATOR}} \psi(n\tau)$$

FLOQUET OPERATORS CAN BE CONSIDERED AS maps

$\psi(n+1) = F\,\psi(n)$, F is unitary!

Under time reversal, there must be an antiunitary operator T, such that

$$T F T^{-1} = F^{-1} = F^{\dagger}$$

DYSON'S CIRCULAR ENSEMBLES

RANDOM, UNITARY MATRICES

Suppose $\quad F|\phi_j\rangle = e^{-i\phi_j}|\phi_j\rangle \quad j=1,\dots N$

Then for RUM

$$P(\{\phi\}) = \frac{1}{N_\beta}\prod_{i<j}\left|e^{-i\phi_i} - e^{-i\phi_j}\right|^{\beta}$$

$$\beta = \begin{cases} 1 & \text{COE} \\ 2 & \text{CUE} \\ 4 & \text{CSE} \end{cases}$$

For Integrable Systems $\quad P(\{\phi\}) = \dfrac{1}{(2\pi)^N}$

Examples :

Kicked Top (Haake)

$$H(t) = \frac{\hbar p}{\tau} J_y + \frac{\hbar k}{2I} J_z^2 \sum \delta(t - n\tau)$$

p, k, τ are all constants

$$\cancel{F = e^{-\frac{i}{\hbar} J_z^2} e^{-\frac{ip}{\hbar} J_y}}$$

$$F_\theta = e^{-\frac{i}{\hbar}\left(\frac{\hbar k}{2I}\right) J_z^2} e^{-\frac{i}{\hbar} \hbar p J_y}$$

$$= e^{-\frac{ik}{2I} J_z^2} e^{-ip J_y}$$

Kicked Rotator

$$H = \frac{L^2}{2} + k \cos\theta \sum \delta(t - n\tau)$$

$$F = e^{-\frac{i}{\hbar} k \cos\theta} e^{-\frac{i}{2\hbar} \tau L^2}$$