DORFMAN LECTURE NOTES, PART 2

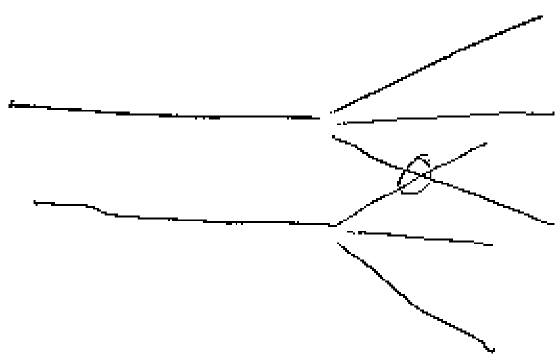
QUANTUM CHAOS, IHP

PARIS, NOVEMBER 2007

LEVEL DYNAMICS, CURVATURE, CALOGERO-SUTHERLAND-MOSER MODEL.

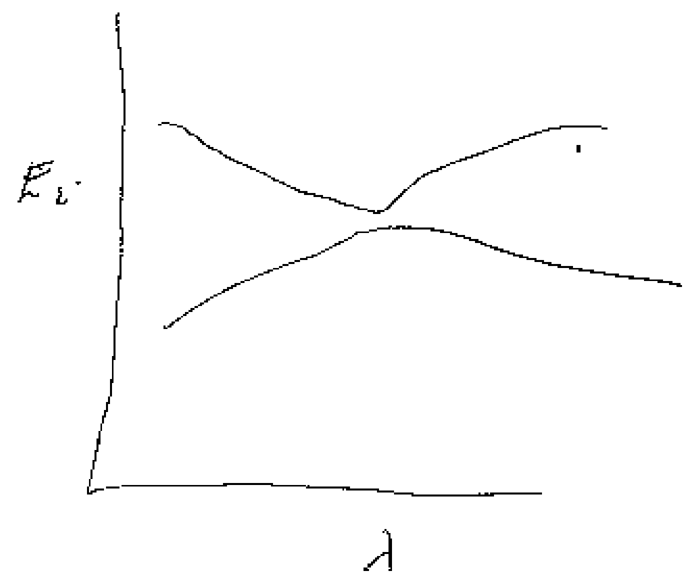
Suppose we have a Hamiltonian with a dependence on a parameter λ, say (magnetic field, etc

H-atom Mag field - Zeemann splitting



← level crossing.

However for CCS - there is level repulsion
See Haake's book.



The "motion" of the levels, as a function of λ is like a system of interacting particles
N levels → N "particles"

As $N \to \infty$, similar to a thermodynamic limit

Calogero-Sutherland-Moser Integrable system

E: QUASI ENERGIES OF ONE-PERIOD PROPAGATOR

$$\hat{H} = A\hat{S}_z^2 - \mu B \hat{S}_x \sum_{-\infty}^{\infty} \delta(t-nT)$$

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$

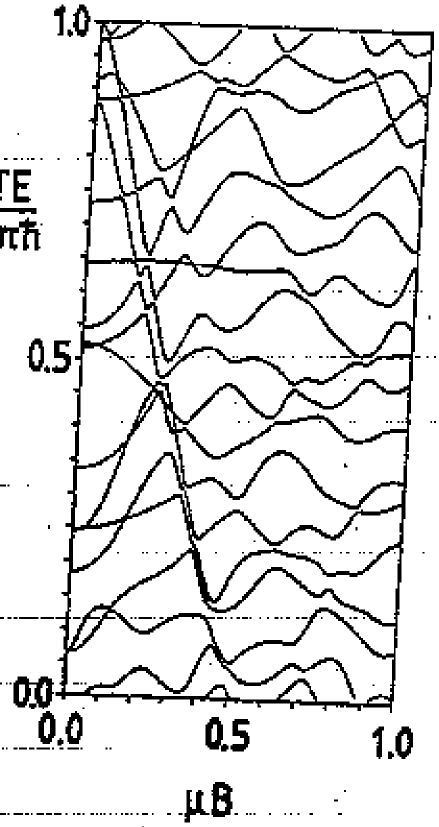


FIG. 1. Quasienergy spectrum vs the external magnetic field B for the pulsed-quantum spin system of Hamiltonian (2.5) with spin magnitude $s=16$, and parameter values $A=\mu=\hbar=1$ and $T=2\pi$. Only the even parity manifold of dimension 17 is depicted in the fundamental zone $(0\leq E<1)$.

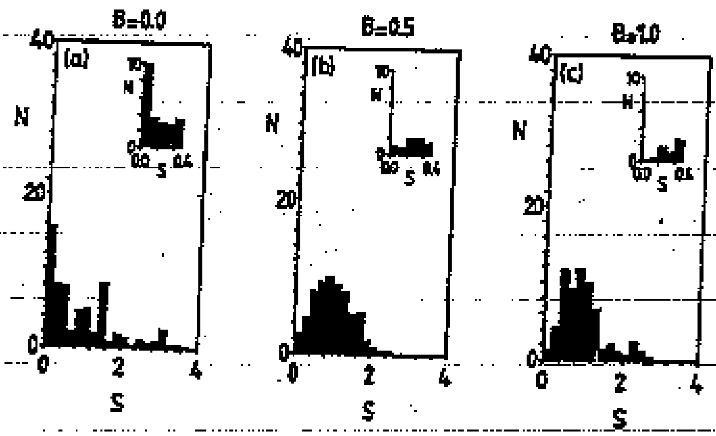


FIG. 2. Density of the level spacing of the same system as in Fig. 1 but with spin magnitude $s=64$ at different magnetic-field strengths: (a) $B=0.0$, (b) $B=0.5$, (c) $B=1.0$. The cells of the histogram have a size $\Delta B=0.2$. Inset: magnification of the histogram near-zero spacing but with fivefold finer cells.

P. GASPARD et al. PHYS. REV. A 42, 4015 (1990)

For H of the form

$$H = H_0 + \lambda V \sum \delta(t - n\tau)$$

We have Floquet operators + quasi energies

The equations of motion of the quasi-energies were developed by Pechukas + Yukawa.

For simple cases that correspond to the dynamics of a known model the Calogero-Moser model of several interacting particles, which can be solved.

Several People used RMT + stat mech to compute the distribution of curvatures, K,

$$P(K) \sim \frac{1}{K^{2+\nu}} \qquad \begin{array}{ll} \nu = 1 & COE \\ \phantom{\nu = {}}2 & CUE \\ \phantom{\nu = {}}4 & CSE \end{array}$$

NEED FIGURE FROM ZC's PAPER

The B-G-S Conjecture

Bohigas - Gianonni - Schmidt

~~For any system whose classical~~ counterpart

For any time-reversal invariant system whose classical counterpart is chaotic (K-system), then the GOE describes the spacing distribution of the energy levels.

(May also apply to ergodic systems weaker than chaotic.)

Also mentioned earlier (1980) by Casati + Co Workers!

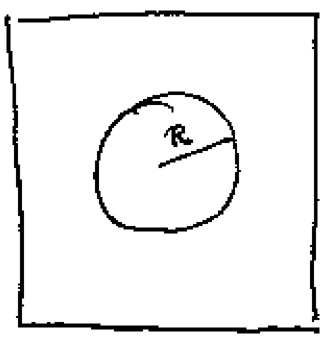
Interesting Counter-examples to RMT.

A. Classically integrable systems
P. Seba PRL 64, 1855, (1990)
Rahav & Fishman Nonlinearity 15, 1541 (2002)
Bogomolny & Giraud 15, 993 (2002)

Rectangle billiard with δ function inside

classically - does not notice the δ-fn

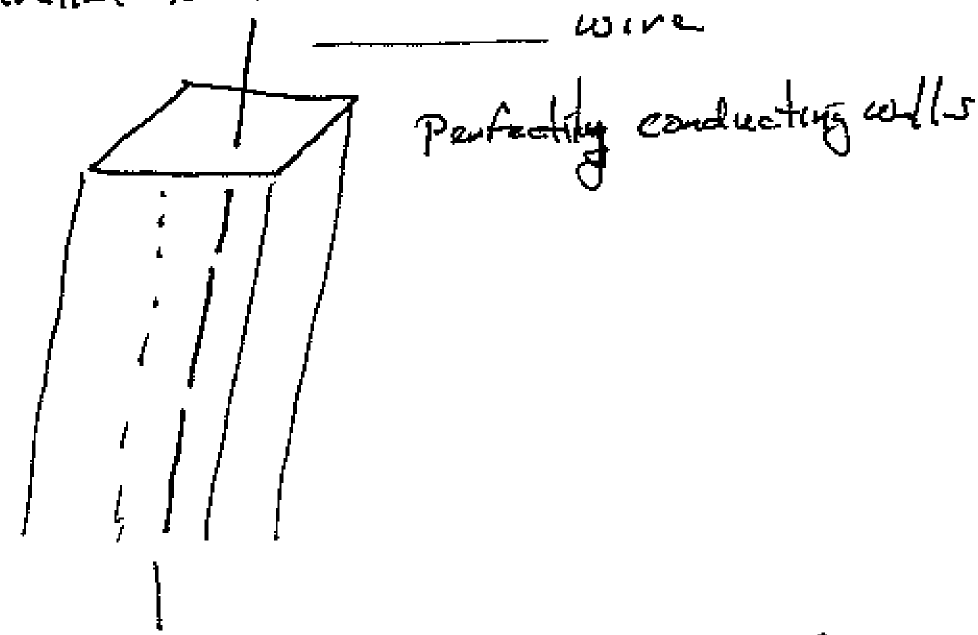
Q.M. It changes things due to diffraction of W. function



Sinai Billiard - classically chaotic
QM GOE Statistics
What happens if $R \to 0$

Point scatterer as $R \to 0$ requires a theory of self-adjoint extensions of an operator which is not self adjoint

Rectangular Waveguide with wire inside
parallel to the sides



Spacing statistics satisfy GOE

~~NEED FIG 1 FROM SEBA FIG 2~~
~~PRL 64, 1855 (1990).~~

Wave function shows messy structure
Heller 53, 1515 (1984)
McDonald + Kaufman PR A 37, 3067 (1988).

NOTE: CAN NEVER MAKE Wave Length
Small compared with size of scatterer
- Classical Limit is problematic.

CHAOTIC SYSTEMS

MOTION ON SURFACE OF CONSTANT NEGATIVE CURVATURE:

Classically Chaotic - 1st Example of Ergodic System

QM. Solutions of Laplace-Beltrami eq. for S.E. on Surface of Const neg curvature

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Gutzwiller's Book

Balazs & Voros Chaos on Pseudosphere
Phys Repts 143, 109 (1986)

Bogomolny, Georgeot, Giannoni, Schmidt
Arithmetical Chaos, Phys Rept
291, 219 (1997).

Don't always get GOE
Due to
Hidden symmetries associated with arith
metical group structure
Hecke operators - infinitely many acting as
pseudo symmetries
Fig 3 BGGS - p 229
6
For tiling "triangles"
Highly non generic.

ARITHMETIC QUANTUM CHAOS

JENS MARKLOF

1. Introduction

The central objective in the study of *quantum chaos* is to characterize universal properties of quantum systems that reflect the regular or chaotic features of the underlying classical dynamics. Most developments of the past 25 years have been influenced by the pioneering models on statistical properties of eigenstates (Berry 1977) and energy levels (Berry and Tabor 1977; Bohigas, Giannoni and Schmit 1984). *Arithmetic quantum chaos* (AQC) refers to the investigation of quantum system with additional arithmetic structures that allow a significantly more extensive analysis than is generally possible. On the other hand, the special number-theoretic features also render these systems non-generic, and thus some of the expected universal phenomena fail to emerge. Important examples of such systems include the modular surface and linear automorphisms of tori ('cat maps') which will be described below.

The geodesic motion of a point particle on a compact Riemannian surface $\mathcal{M}$ of constant negative curvature is the prime example of an Anosov flow, one of the strongest characterizations of dynamical chaos. The corresponding quantum eigenstates φ_j and energy levels λ_j are given by the solution of the eigenvalue problem for the Laplace-Beltrami operator Δ (or *Laplacian* for short)

$$(\Delta + \lambda)\varphi = 0, \quad \|\varphi\|_{L^2(\mathcal{M})} = 1, \tag{1}$$

where the eigenvalues

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \tag{2}$$

form a discrete spectrum with an asymptotic density governed by Weyl's law

$$\#\{j : \lambda_j \leq \lambda\} \sim \frac{\operatorname{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda, \qquad \lambda \to \infty. \tag{3}$$

We rescale the sequence by setting

$$X_j = \frac{\operatorname{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda_j \tag{4}$$

which yields a sequence of asymptotic density one. One of the central conjectures in AQC says that, if $\mathcal{M}$ is an *arithmetic* hyperbolic surface (see Sec. 2 for examples of this very special class of surfaces of constant negative curvature), the *eigenvalues* of the Laplacian have the same

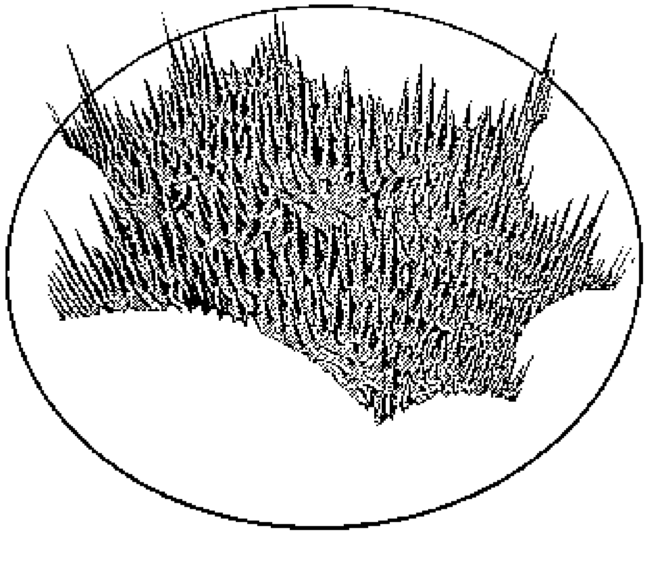

Figure 1. Image of the absolute value squared of an eigenfunction $\varphi_j(z)$ for a non-arithmetic surface of genus two. The surface is obtained by identifying opposite sides of the fundamental region [1].

local statistical properties as independent random variables from a Poisson process, see e.g. the surveys [12, 3]. This means that the probability of finding k eigenvalues X_j in randomly shifted interval $[X, X+L]$ of fixed length L is distributed according to the Poisson law $L^k e^{-L}/k!$. The gaps between eigenvalues have an exponential distribution,

$$\frac{1}{N}\#\{j \leq N : X_{j+1} - X_j \in [a,b]\} \to \int_a^b e^{-s}\, ds \tag{5}$$

as $N \to \infty$, and thus eigenvalues are likely to appear in clusters. This is in contrast to the general expectation that the energy level statistics of *generic* chaotic systems follow the distributions of random matrix ensembles; Poisson statistics are usually associated with quantized integrable systems. Although we are at present far from a proof of (5), the deviation from random matrix theory is well understood, see Sec. 3.

Highly excited quantum eigenstates φ_j ($j \to \infty$), cf. Fig. 1, of chaotic systems are conjectured to behave locally like random wave solutions of (1), where boundary conditions are ignored. This hypothesis was put forward by Berry in 1977 and tested numerically, e.g., in the case of certain arithmetic and non-arithmetic surfaces of constant negative curvature [6, 1]. One of the implications is that eigenstates should have uniform mass on the surface $\mathcal{M}$, i.e., for any bounded continuous function $g : \mathcal{M} \to \mathbb{R}$

$$\int_{\mathcal{M}} |\varphi_j|^2 g\, dA \to \int_{\mathcal{M}} g\, dA, \qquad j \to \infty, \tag{6}$$

Date: September 13, 2005.

The author is supported by an EPSRC Advanced Research Fellowship. Fig. 1 is reproduced by courtesy of R. Aurich, University of Ulm.

Prepared for the Encyclopedia of Mathematical Physics.

where dA is the Riemannian area element on $\mathcal{M}$. This phenomenon, referred to as *quantum unique ergodicity* (QUE), is expected to hold for general surfaces of negative curvature, according to a conjecture by Rudnick and Sarnak (1994). In the case of arithmetic hyperbolic surfaces, there has been substantial progress on this conjecture in the works of Lindenstrauss, Watson and Luo-Sarnak, see Secs. 5, 6 and 7, as well as the review [13]. For general manifolds with ergodic geodesic flow, the convergence in (6) is so far established only for subsequences of eigenfunctions of density one (Schnirelman-Zelditch-Colin de Verdière Theorem, cf. [14]), and it cannot be ruled out that exceptional subsequences of eigenfunctions have singular limit, e.g., localized on closed geodesics. Such 'scarring' of eigenfunctions, at least in some weak form, has been suggested by numerical experiments in Euclidean domains, and the existence of singular quantum limits is a matter of controversy in the current physics and mathematics literature. A first rigorous proof of the existence of scarred eigenstates has recently been established in the case of quantized toral automorphisms. Remarkably, these quantum cat maps may also exhibit quantum unique ergodicity. A more detailed account of results for these maps is given in Sec. 8, see also [9, 5].

There have been a number of other fruitful interactions between quantum chaos and number theory, in particular the connections of spectral statistics of integrable quantum systems with the value distribution properties of quadratic forms, and analogies in the statistical behaviour of energy levels of chaotic systems and the zeros of the Riemann zeta function. We refer the reader to [8] and [2], respectively, for information on these topics.

2. Hyperbolic surfaces

Let us begin with some basic notions of hyperbolic geometry. The hyperbolic plane $\mathbb{H}$ may be abstractly defined as the simply connected two-dimensional Riemannian manifold with Gaussian curvature -1. A convenient parametrization of $\mathbb{H}$ is provided by the complex upper half plane, $\mathfrak{H} = \{x + iy : x \in \mathbb{R},\ y > 0\}$, with Riemannian line and volume elements

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \qquad dA = \frac{dx\, dy}{y^2}, \tag{7}$$

respectively. The group of orientation-preserving isometries of $\mathbb{H}$ is given by fractional linear transformations

$$\mathfrak{H} \to \mathfrak{H}, \qquad z \mapsto \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \tag{8}$$

where $\mathrm{SL}(2,\mathbb{R})$ is the group of 2×2 matrices with unit determinant. Since the matrices 1 and -1 represent the same transformation, the group of orientation-preserving isometries can be identified with $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. A finite-volume hyperbolic surface may now be represented as the quotient $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group of the first kind. An *arithmetic* hyperbolic surface

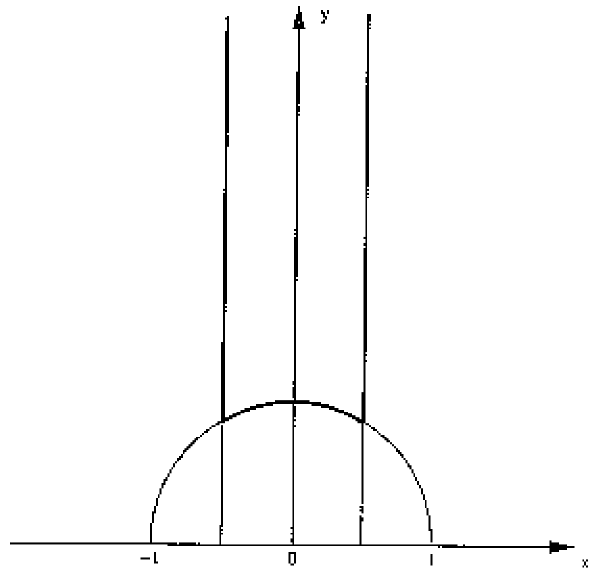

Figure 2. Fundamental domain of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ in the complex upper half plane.

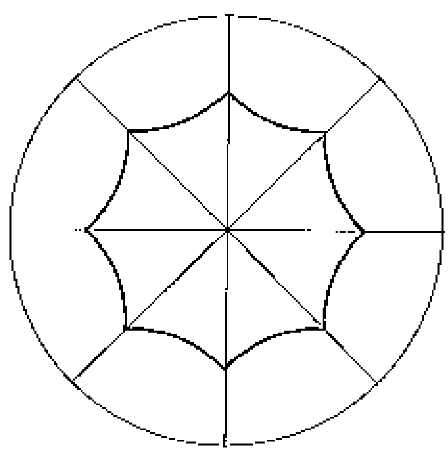

Figure 3. Fundamental domain of the regular octagon in the Poincaré disc.

(such as the modular surface) is obtained, if Γ has, loosely speaking, some representation in $n \times n$ matrices with integer coefficients, for some suitable n. This is evident in the case of the modular surface, where the fundamental group is the *modular group* $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{PSL}(2,\mathbb{R}) : a,b,c,d \in \mathbb{Z}\}/\{\pm 1\}$.

A fundamental domain for the action of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathfrak{H}$ is the set

$$\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})} = \left\{ z \in \mathfrak{H} : |z| > 1, -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2} \right\}, \tag{9}$$

see fig. 2. The modular group is generated by the translation $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) : z \mapsto z+1$ and the inversion $(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) : z \mapsto -1/z$. These generators identify sections of the boundary of $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$. By gluing the fundamental domain along identified edges we obtain a realization of the modular surface, a non-compact surface with one cusp at $z \to \infty$, and two conic singularities at $z = \mathrm{i}$ and $z = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$.

An interesting example of a *compact* arithmetic surface is the 'regular octagon', a hyperbolic surface of genus two. Its fundamental domain is shown in fig. 3 as a subset of the Poincaré disc $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$, which yields an alternative parametrization of the hyperbolic plane $\mathbb{H}$. In these coordinates, the Riemannian line and volume element read

$$(10) \qquad ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}, \qquad dA = \frac{4\,dx\,dy}{(1-x^2-y^2)^2}.$$

The group of orientation-preserving isometries is now represented by $\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm 1\}$, where

$$(11) \quad \mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\},$$

acting on $\mathfrak{D}$ as above via fractional linear transformations. The fundamental group of the regular octagon surface is the subgroup of all elements in $\mathrm{PSU}(1,1)$ with coefficients of the form

$$(12) \qquad \alpha = k + l\sqrt{2}, \qquad \beta = (m + n\sqrt{2})\sqrt{1+\sqrt{2}},$$

where $k, l, m, n \in \mathbb{Z}[\mathrm{i}]$, i.e., Gaussian integers of the form $k_1 + \mathrm{i}k_2$, $k_1, k_2 \in \mathbb{Z}$. Note that not all choices of $k, l, m, n \in \mathbb{Z}[\mathrm{i}]$ satisfy the condition $|\alpha|^2 - |\beta|^2 = 1$. Since all elements $\gamma \neq 1$ of Γ act fix-point free on $\mathbb{H}$, the surface $\Gamma\backslash\mathbb{H}$ is smooth without conic singularities.

In the following we will restrict our attention to a representative case, the modular surface with $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$.

3. Eigenvalue statistics and Selberg trace formula

The statistical properties of the rescaled eigenvalues X_j (cf. (4)) of the Laplacian can be characterized by their distribution in small intervals

$$(13) \qquad \mathcal{N}(x, L) := \#\{j : x \le X_j \le x + L\},$$

where x is uniformly distributed, say, in the interval $[X, 2X]$, X large. Numerical experiments by Bogomolny, Georgeot, Giannoni and Schmit, as well as Bolte, Steil and Steiner (see refs. in [3]) suggest that the X_j are asymptotically Poisson distributed:

Conjecture 1. *For any bounded function $g : \mathbb{Z}_{\ge 0} \to \mathbb{C}$ we have*

$$(14) \qquad \frac{1}{X}\int_X^{2X} g(\mathcal{N}(x,L))\,dx \to \sum_{k=0}^{\infty} g(k)\frac{L^k e^{-L}}{k!},$$

as $T \to \infty$.

One may also consider larger intervals, where $L \to \infty$ as $X \to \infty$. In this case the assumption on the independence of the X_j predicts a central limit theorem. Weyl's law (3) implies that the expectation value is asymptotically, for $T \to \infty$,

$$(15) \qquad \frac{1}{X}\int_X^{2X} \mathcal{N}(x,L)\,dx \sim L.$$

This asymptotics holds for any sequence of L bounded away from zero (e.g. L constant, or $L \to \infty$).

Define the variance by

$$(16) \qquad \Sigma^2(X,L) = \frac{1}{X}\int_X^{2X} \left(\mathcal{N}(x,L) - L\right)^2 dx.$$

In view of the above conjecture, one expects $\Sigma^2(X,L) \sim L$ in the limit $X \to \infty$, $L/\sqrt{X} \to 0$ (the variance exhibits a less universal behaviour in the range $L \gg \sqrt{X}$,[1] cf. [12]), and a central limit theorem for the fluctuations around the mean:

Conjecture 2. *For any bounded function $g : \mathbb{R} \to \mathbb{C}$ we have*

(17)

$$\frac{1}{X}\int_X^{2X} g\left(\frac{\mathcal{N}(x,L) - L}{\sqrt{\Sigma^2(x,L)}}\right) dx \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-\frac{1}{2}t^2}dt,$$

as $X, L \to \infty$, $L \ll X$.

The main tool in the attempts to prove the above conjectures has been the Selberg trace formula. It relates sums over eigenvalues of the Laplacians to sums over lengths of closed geodesics on the hyperbolic surface. The trace formula is in its simplest form in the case of compact hyperbolic surfaces; we have

$$(18) \qquad \sum_{j=0}^{\infty} h(\rho_j) = \frac{\mathrm{Area}(\mathcal{M})}{4\pi}\int_{-\infty}^{\infty} h(\rho)\tanh(\pi\rho)\,\rho\,d\rho + \sum_{\gamma \in H_*}\sum_{n=1}^{\infty} \frac{\ell_\gamma\, g(n\ell_\gamma)}{2\sinh(n\ell_\gamma/2)},$$

where H_* is the set of all primitive oriented closed geodesics γ, and ℓ_γ their lengths. The quantity ρ_j is related to the eigenvalue λ_j by the equation $\lambda_j = \rho_j^2 + \frac{1}{4}$. The trace formula (18) holds for a large class of even test functions h. E.g. it is sufficient to assume that h is infinitely differentiable, and that the Fourier transform of h,

$$(19) \qquad g(t) = \frac{1}{2\pi}\int_{\mathbb{R}} h(\rho)\,e^{-\mathrm{i}\rho t}d\rho,$$

has compact support. The trace formula for non-compact surfaces has additional terms from the parabolic elements in the corresponding group, and includes also sums over the resonances of the continuous part of the spectrum. The non-compact modular surface behaves in many ways like a compact surface. In particular, Selberg showed that the number of eigenvalues embedded in the continuous spectrum satisfies the same Weyl law as in the compact case [13].

Setting

$$(20) \qquad h(\rho) = \chi_{[X,X+L]}\left(\tfrac{\mathrm{Area}(\mathcal{M})}{4\pi}(\rho^2 + \tfrac{1}{4})\right),$$

where $\chi_{[X,X+L]}$ is the characteristic function of the interval $[X, X+L]$, we may thus view $\mathcal{N}(X,L)$ as the left-hand side

[1] The notation $A \ll B$ means there is a constant $c > 0$ such that $A \le cB$

of the trace formula. The above test function h is however not admissible, and requires appropriate smoothing. Luo and Sarnak (cf. [13]) developed an argument of this type to obtain a lower bound on the average number variance,

$$\frac{1}{L}\int_0^L \Sigma^2(X,L')dL' \gg \frac{\sqrt{X}}{(\log X)^2} \tag{21}$$

in the regime $\sqrt{X}/\log X \ll L \ll \sqrt{X}$, which is consistent with the Poisson conjecture $\Sigma^2(X,L)\sim L$. Bogomolny, Levyraz and Schmit suggested a remarkable limiting formula for the two-point correlation function for the modular surface (cf. [3, 4]), based on an analysis of the correlations between multiplicities of lengths of closed geodesics. A rigorous analysis of the fluctuations of multiplicities is given by Peter, cf. [4]. Rudnick [10] has recently established a smoothed version of Conjecture 2 in the regime

$$\frac{\sqrt{X}}{L}\to\infty, \quad \frac{\sqrt{X}}{L\log X}\to 0, \tag{22}$$

where the characteristic function in (20) is replaced by a certain class of smooth test functions.

All of the above approaches use the Selberg trace formula, exploiting the particular properties of the distribution of lengths of closed geodesics in arithmetic hyperbolic surfaces. These will be discussed in more detail in the next section, following the work of Bogomolny, Georgeot, Giannoni and Schmit, Bolte, and Luo and Sarnak, see [3, 12] for references.

4. Distribution of lengths of closed geodesics

The classical prime geodesic theorem asserts that the number $N(\ell)$ of primitive closed geodesics of length less than ℓ is asymptotically

$$N(\ell)\sim\frac{e^\ell}{\ell}. \tag{23}$$

One of the significant geometrical characteristics of arithmetic hyperbolic surfaces is that the number of closed geodesics with the same length ℓ grows exponentially with ℓ. This phenomenon is easiest explained in the case of the modular surface, where the set of lengths ℓ appearing in the lengths spectrum is characterized by the condition

$$2\cosh(\ell/2) = |\operatorname{tr}\gamma| \tag{24}$$

where γ runs over all elements in $SL(2,\mathbb{Z})$ with $|\operatorname{tr}\gamma|>2$. It is not hard to see that any integer $n>2$ appears in the set $\{|\operatorname{tr}\gamma| : \gamma\in SL(2,\mathbb{Z})\}$, and hence the set of distinct lengths of closed geodesics is

$$\mathcal{L} = \{2\operatorname{arcosh}(n/2) : n=3,4,5,\ldots\}. \tag{25}$$

The number of distinct lengths less than ℓ is therefore asymptotically for large ℓ

$$N'(\ell) = \#(\mathcal{L}\cap[0,\ell]) \sim e^{\ell/2}. \tag{26}$$

Eqs. (26) and (23) say that *on average* the number of geodesics with the same lengths is at least $\asymp e^{\ell/2}/\ell$.

The prime geodesic theorem (23) holds equally for all hyperbolic surfaces with finite area, while (26) is specific to the modular surface. For general arithmetic surfaces, we have the upper bound

$$N'(\ell)\le ce^{\ell/2} \tag{27}$$

for some constant $c>0$ that may depend on the surface. Although one expects $N'(\ell)$ to be asymptotic to $\frac{1}{2}N(\ell)$ for generic surfaces (since most geodesics have a time-reversal partner which thus has the same length, and otherwise all lengths are distinct), there are examples of non-arithmetic Hecke triangles where numerical and heuristic arguments suggest $N'(\ell)\sim c_1 e^{c_2\ell}/\ell$ for suitable constants $c_1>0$ and $0<c_2<1/2$, cf. [4]. Hence exponential degeneracy in the length spectrum seems to occur in a weaker form also for non-arithmetic surfaces.

A further useful property of the length spectrum of arithmetic surfaces is the *bounded clustering property*: there is a constant C (again surface dependent) such that

$$\#(\mathcal{L}\cap[\ell,\ell+1])\le C \tag{28}$$

for all ℓ. This fact is evident in the case of the modular surface; the general case is proved by Luo and Sarnak, cf. [12].

5. Quantum unique ergodicity

The unit tangent bundle of a hyperbolic surface $\Gamma\backslash\mathbb{H}$ describes the physical phase space on which the classical dynamics takes place. A convenient parametrization of the unit tangent bundle is given by the quotient $\Gamma\backslash PSL(2,\mathbb{R})$—this may be seen be means of the Iwasawa decomposition for an element $g\in PSL(2,\mathbb{R})$,

$$g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\theta/2 & \sin\theta/2\\ -\sin\theta/2 & \cos\theta/2\end{pmatrix}, \tag{29}$$

where $x+iy\in\mathfrak{H}$ represents the position of the particle in $\Gamma\backslash\mathbb{H}$ in half-plane coordinates, and $\theta\in[0,2\pi)$ the direction of its velocity. Multiplying the matrix (29) from the left by $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and writing the result again in the Iwasawa form (29), one obtains the action

$$(z,\phi)\mapsto\left(\frac{az+b}{cz+d},\theta-2\arg(cz+d)\right), \tag{30}$$

which represents precisely the geometric action of isometries on the unit tangent bundle.

The geodesic flow Φ^t on $\Gamma\backslash PSL(2,\mathbb{R})$ is represented by the right translation

$$\Phi^t:\Gamma g\mapsto\Gamma g\begin{pmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix}. \tag{31}$$

The Haar measure μ on $PSL(2,\mathbb{R})$ is thus trivially invariant under the geodesic flow. It is well known that μ is not the only invariant measure, i.e., Φ^t is not uniquely ergodic, and that there is in fact an abundance of invariant measures. The simplest examples are those with uniform mass on one, or a countable collection of, closed geodesics.

To test the distribution of an eigenfunction φ_j in phase space, one associates with a function $a \in C^\infty(\Gamma\backslash \mathrm{PSL}(2,\mathbb{R}))$ the quantum observable $\mathrm{Op}(a)$, a zeroth order pseudo-differential operator with principal symbol a. Using semi-classical techniques based on Friedrichs symmetrization, on can show that the matrix element

$$\nu_j(a) = \langle \mathrm{Op}(a)\varphi_j, \varphi_j\rangle \tag{32}$$

is asymptotic (as $j \to \infty$) to a positive functional that defines a probability measure on $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$. Therefore, if $\mathcal{M}$ is compact, any weak limit of ν_j represents a probability measure on $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$. Egorov's theorem (cf. [14]) in turn implies that any such limit must be invariant under the geodesic flow, and the main challenge in proving quantum unique ergodicity is to rule out all invariant measures apart from Haar.

Conjecture 3 (Rudnick and Sarnak (1994) [12, 13]). *For every compact hyperbolic surface $\Gamma\backslash\mathbb{H}$, the sequence ν_j converges weakly to μ.*

Lindenstrauss has proved this conjecture for compact arithmetic hyperbolic surfaces of congruence type (such as the second example in Sec. 2) for special bases of eigenfunctions, using ergodic-theoretic methods. These will be discussed in more detail in Sec. 6 below. His results extend to the non-compact case, i.e., to the modular surface where $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. Here he shows that any weak limit of subsequences of ν_j is of the form $c\mu$, where c is a constant with values in $[0,1]$. One believes that $c = 1$, but with present techniques it cannot be ruled out that a proportion of the mass of the eigenfunction escapes into the non-compact cusp of the surface. For the modular surface, $c = 1$ can be proved under the assumption of the generalized Riemann hypothesis, see Sec. 7 and [13]. Quantum unique ergodicity also holds for the continuous part of the spectrum, which is furnished by the Eisenstein series $E(z,s)$, where $s = 1/2+ir$ is the spectral parameter. Note that the measures associated with the matrix elements

$$\nu_r(a) = \langle \mathrm{Op}(a)E(\,\cdot\,,1/2+ir), E(\,\cdot\,,1/2+ir)\rangle \tag{33}$$

are not probability but only Radon measures, since $E(z,s)$ is not square-integrable. Luo and Sarnak, and Jakobson have shown that

$$\lim_{r\to\infty} \frac{\nu_r(a)}{\nu_r(b)} = \frac{\mu(a)}{\mu(b)} \tag{34}$$

for suitable test functions $a, b \in C^\infty(\Gamma\backslash \mathrm{PSL}(2,\mathbb{R}))$, cf. [13].

6. Hecke operators, entropy and measure rigidity

For compact surfaces, the sequence of probability measures approaching the matrix elements ν_j is relatively compact. That is, every infinite sequence contains a convergent subsequence. Lindenstrauss' central idea in the proof of quantum unique ergodicity is to exploit the presence of Hecke operators to understand the invariance properties of possible quantum limits. We will sketch his argument in the case of the modular surface (ignoring issues related to the non-compactness of the surface), where it is most transparent.

For every positive integer n, the Hecke operator T_n acting on continuous functions on $\Gamma\backslash\mathbb{H}$ with $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is defined by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{a,d=1\\ ad=n}}^{n} \sum_{b=0}^{d-1} f\Big(\frac{az+b}{d}\Big). \tag{35}$$

The set M_n of matrices with integer coefficients and determinant n can be expressed as the disjoint union

$$M_n = \bigcup_{\substack{a,d=1\\ ad=n}}^{n} \bigcup_{b=0}^{d-1} \Gamma \begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \tag{36}$$

and hence the sum in (35) can be viewed as a sum over the cosets in this decomposition. We note the product formula

$$T_m T_n = \sum_{d|\gcd(m,n)} T_{mn/d^2}. \tag{37}$$

The Hecke operators are normal, form a commuting family, and in addition commute with the Laplacian Δ. In the following we consider an orthonormal basis of eigenfunctions φ_j of Δ that are simultaneously eigenfunctions of all Hecke operators. We will refer to such eigenfunctions as *Hecke eigenfunctions*. The above assumption is automatically satisfied, if the spectrum of Δ is simple (i.e. no eigenvalues coincide), a property conjectured by Cartier and supported by numerical computations. Lindenstrauss' work is based on the following two observations. Firstly, all quantum limits of Hecke eigenfunctions are geodesic-flow invariant measures of positive entropy, and secondly, the only such measure of positive entropy that is recurrent under Hecke correspondences is Lebesgue measure.

The first property is proved by Bourgain and Lindenstrauss (2003) and refines arguments of Rudnick and Sarnak (1994) and Wolpert (2001) on the distribution of Hecke points (see [13] for references to these papers). For a given point $z \in \mathbb{H}$ the set of Hecke points is defined as

$$T_n(z) := M_n z. \tag{38}$$

For most primes, the set $T_{p^k}(z)$ comprises $(p+1)p^{k-1}$ distinct points on $\Gamma\backslash\mathbb{H}$. For each z, the Hecke operator T_n may now be interpreted as the adjacency matrix for a finite graph embedded in $\Gamma\backslash\mathbb{H}$, whose vertices are the Hecke points $T_n(z)$. Hecke eigenfunctions φ_j with

$$T_n\varphi_j = \lambda_j(n)\varphi_j \tag{39}$$

give rise to eigenfunctions of the adjacency matrix. Exploiting this fact, Bourgain and Lindenstrauss show that for a large set of integers n

$$|\varphi_j(z)|^2 \ll \sum_{w\in T_n(z)} |\varphi_j(w)|^2, \tag{40}$$

i.e. pointwise values of $|\varphi_j|^2$ cannot be substantially larger than its sum over Hecke points. This and the observation that Hecke points for a large set of integers n are sufficiently uniformly distributed on $\Gamma\backslash\mathbb{H}$ as $n\to\infty$, yields the estimate of positive entropy with a quantitative lower bound.

Lindenstrauss' proof of the second property, which shows that Lebesgue measure is the only quantum limit of Hecke eigenfunctions, is a result of a currently very active branch of ergodic theory: measure rigidity. Invariance under the geodesic flow alone is not sufficient to rule out other possible limit measures. In fact there are uncountably many measures with this property. As limits of Hecke eigenfunctions, all quantum limits possess an additional property, namely recurrence under Hecke correspondences. Since the explanation of these is rather involved, let us recall an analogous result in a simpler set-up. The map $\times 2 : x\mapsto 2x \bmod 1$ defines a hyperbolic dynamical system on the unit circle with a wealth of invariant measures, similar to the case of the geodesic flow on a surface of negative curvature. Furstenberg conjectured that, up to trivial invariant measures that are localized on finitely many rational points, Lebesgue measure is the only $\times 2$-invariant measure that is also invariant under action of $\times 3 : x\mapsto 3x \bmod 1$. This fundamental problem is still unsolved and one of the central conjectures in measure rigidity. Rudolph however showed that Furstenberg's conjecture is true if one restricts the statement to $\times 2$-invariant measures of positive entropy, cf. [11]. In Lindenstrauss' work, $\times 2$ plays the role of the geodesic flow, and $\times 3$ the role of the Hecke correspondences. Although it might also here be interesting to ask whether and analogue of Furstenberg's conjecture holds, it is inessential for the proof of QUE due to the positive entropy of quantum limits discussed in the previous paragraph.

7. Eigenfunctions and L-functions

An even eigenfunction $\varphi_j(z)$ for $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ has the Fourier expansion

$$\varphi_j(z)=\sum_{n=1}^{\infty} a_j(n) y^{1/2} K_{i\rho_j}(2\pi n y)\cos(2\pi n x). \tag{41}$$

We associate with $\varphi_j(z)$ the Dirichlet series

$$L(s,\varphi_j)=\sum_{n=1}^{\infty} a_j(n) n^{-s} \tag{42}$$

which converges for $\operatorname{Re} s$ large enough. These series have an analytic continuation to the entire complex plane $\mathbb{C}$ and satisfy a functional equation,

$$\Lambda(s,\varphi_j)=\Lambda(1-s,\varphi_j) \tag{43}$$

where

$$\Lambda(s,\varphi_j)=\pi^{-s}\Gamma\big(\tfrac{s+i\rho_j}{2}\big)\Gamma\big(\tfrac{s-i\rho_j}{2}\big)L(s,\varphi_j). \tag{44}$$

If $\varphi_j(z)$ is in addition an eigenfunction of all Hecke operators, then the Fourier coefficients in fact coincide (up to a normalization constant) with the eigenvalues of the Hecke operators

$$a_j(m)=\lambda_j(m)a_j(1). \tag{45}$$

If we normalize $a_j(1)=1$, the Hecke relations (37) result in an Euler product formula for the L-function,

$$L(s,\varphi_j)=\prod_{p \text{ prime}}\big(1-a_j(p)p^{-s}+p^{-1-2s}\big)^{-1}. \tag{46}$$

These L-functions behave in many other ways like the Riemann zeta or classical Dirichlet L-functions. In particular, they are expected to satisfy a Riemann hypothesis, i.e. all non-trivial zeros are constrained to the critical line $\operatorname{Im} s = 1/2$.

Questions on the distribution of Hecke eigenfunctions, such as quantum unique ergodicity or value distribution properties, can now be translated to analytic properties of L-functions. We will discuss two examples.

The asymptotics in (6) can be established by proving (6) for the choices $g=\varphi_k$, $k=1,2,\ldots$, i.e.,

$$\int_{\mathcal{M}}|\varphi_j|^2\varphi_k\,dA\to 0. \tag{47}$$

Watson discovered the remarkable relation [13]

$$\left|\int_{\mathcal{M}}\varphi_{j_1}\varphi_{j_2}\varphi_{j_3}\,dA\right|^2 = \frac{\pi^4\Lambda(\frac12,\varphi_{j_1}\times\varphi_{j_2}\times\varphi_{j_3})}{\Lambda(1,\mathrm{sym}^2\varphi_{j_1})\Lambda(1,\mathrm{sym}^2\varphi_{j_2})\Lambda(1,\mathrm{sym}^2\varphi_{j_3})}. \tag{48}$$

The L-functions $\Lambda(s,g)$ in Watson's formula are more advanced cousins of those introduced earlier, see [13] for details. The Riemann hypothesis for such L-functions implies then via (48) a precise rate of convergence to QUE for the modular surface,

$$\int_{\mathcal{M}}|\varphi_j|^2 g\,dA=\int_{\mathcal{M}} g\,dA+O(\lambda_j^{-1/4+\epsilon}), \tag{49}$$

for any $\epsilon>0$, where the implied constant depends on ϵ and g.

A second example on the connection between statistical properties of the matrix elements $\nu_j(a)=\langle \mathrm{Op}(a)\varphi_j,\varphi_j\rangle$ (for fixed a and random j) and values L-functions has appeared in the work of Luo and Sarnak, cf. [13]. Define the variance

$$V_\lambda(a)=\frac{1}{N(\lambda)}\sum_{\lambda_j\le\lambda}|\nu_j(a)-\mu(a)|^2, \tag{50}$$

with $N(\lambda)=\#\{j:\lambda_j\le\lambda\}$; cf. (3). Following a conjecture by Feingold-Peres and Eckhardt et al. (see [13] for references) for 'generic' quantum chaotic systems, one expects a central limit theorem for the statistical fluctuations of the $\nu_j(a)$, where the normalized variance $N(\lambda)^{1/2}V_\lambda(a)$ is asymptotic to the classical autocorrelation function $C(a)$, see eq. (54).

Conjecture 4. *For any bounded function $g : \mathbb{R} \to \mathbb{C}$ we have*

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} g\left(\frac{\nu_j(a) - \mu(a)}{\sqrt{V_\lambda(a)}}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-\frac{1}{2}t^2} dt, \tag{51}$$

as $\lambda \to \infty$.

Luo and Sarnak prove that in the case of the modular surface the variance has the asymptotics

$$\lim_{\lambda \to \infty} N(\lambda)^{1/2} V_\lambda(a) = \langle Ba, a\rangle, \tag{52}$$

where B is a non-negative self-adjoint operator which commutes with the Laplacian Δ and all Hecke operators T_n. In particular we have

$$B\varphi_j = \frac{1}{2} L(\tfrac{1}{2}, \varphi_j) C(\varphi_j)\, \varphi_j, \tag{53}$$

where

$$C(a) := \int_{\mathbb{R}} \int_{\Gamma \backslash \mathrm{PSL}(2,\mathbb{R})} a(\Phi^t(g)) \overline{a(g)} d\mu(g)\, dt \tag{54}$$

is the classical autocorrelation function for the geodesic flow with respect to the observable a [13]. Up to the arithmetic factor $\frac{1}{2} L(\frac{1}{2}, \varphi_j)$, eq. (53) is consistent with the Feingold-Peres prediction for the variance of generic chaotic systems. Furthermore, recent estimates of moments by Rudnick and Soundararajan (2005) indicate that Conjecture 4 is not valid in the case of the modular surface.

8. Quantum eigenstates of cat maps

Cat maps are probably the simplest area-preserving maps on a compact surface that are highly chaotic. They are defined as linear automorphisms on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\Phi_A : \mathbb{T}^2 \to \mathbb{T}^2, \tag{55}$$

where a point $\xi \in \mathbb{R}^2$ (mod $\mathbb{Z}^2$) is mapped to $A\xi$ (mod $\mathbb{Z}^2$); A is a fixed matrix in $\mathrm{GL}(2,\mathbb{Z})$ with eigenvalues off the unit circle (this guarantees hyperbolicity). We view the torus $\mathbb{T}^2$ as a symplectic manifold, the 'phase space' of the dynamical system. Since $\mathbb{T}^2$ is compact, the Hilbert space of quantum states is an N dimensional vector space $\mathcal{H}_N$, N integer. The semiclassical limit, or limit of small wavelengths, corresponds here to $N \to \infty$.

It is convenient to identify $\mathcal{H}_N$ with $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z})$, with inner product

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{N} \sum_{Q \bmod N} \psi_1(Q) \overline{\psi_2(Q)}. \tag{56}$$

For any smooth function $f \in \mathrm{C}^\infty(\mathbb{T}^2)$, define a *quantum observable*

$$\mathrm{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \widehat{f}(n) T_N(n)$$

where $\widehat{f}(n)$ are the Fourier coefficients of f, and $T_N(n)$ are translation operators

$$T_N(n) = e^{\pi i n_1 n_2/N}\, t_2^{n_2} t_1^{n_1}, \tag{57}$$

$$[t_1\psi](Q) = \psi(Q+1), \quad [t_2\psi](Q) = e^{2\pi i Q/N}\, \psi(Q). \tag{58}$$

The operators $\mathrm{Op}_N(a)$ are the analogues of the pseudo-differential operators discussed in Sec. 5.

A quantization of Φ_A is a unitary operator $U_N(A)$ on $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z})$ satisfying the equation

$$U_N(A)^{-1} \mathrm{Op}_N(f) U_N(A) = \mathrm{Op}_N(f \circ \Phi_A) \tag{59}$$

for all $f \in \mathrm{C}^\infty(\mathbb{T}^2)$. There are explicit formulas for $U_N(A)$ when A is in the group

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : ab \equiv cd \equiv 0 \bmod 2 \right\}; \tag{60}$$

these may be viewed as analogues of the Shale-Weil or metaplectic representation for $\mathrm{SL}(2)$. E.g., the quantization of

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \tag{61}$$

yields

$$U_N(A)\psi(Q) = N^{-\frac{1}{2}} \sum_{Q' \bmod N} \exp\left[\frac{2\pi i}{N}(Q^2 - QQ' + Q'^2)\right] \psi(Q'). \tag{62}$$

In analogy with (1) we are interested in the statistical features of the eigenvalues and eigenfunctions of $U_N(A)$, i.e., the solutions to

$$U_N(A)\varphi = \lambda\varphi, \quad \|\varphi\|_{\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z})} = 1. \tag{63}$$

Unlike typical quantum-chaotic maps, the statistics of the N eigenvalues

$$\lambda_{N1}, \lambda_{N2}, \ldots, \lambda_{NN} \in \mathrm{S}^1 \tag{64}$$

do not follow the distributions of unitary random matrices in the limit $N \to \infty$, but are rather singular [7]. In analogy with the Selberg trace formula for hyperbolic surfaces (18), there is an exact trace formula relating sums over eigenvalues of $U_N(A)$ with sums over fixed points of the classical map [7].

As in the case of arithmetic surfaces, the *eigenfunctions* of cat maps appear to behave more generically. The analogue of the Schnirelman-Zelditch-Colin de Verdière Theorem states that, for any orthonormal basis of eigenfunctions $\{\varphi_{Nj}\}_{j=1}^N$ we have, for all $f \in \mathrm{C}^\infty(\mathbb{T}^2)$,

$$\langle \mathrm{Op}(f)\varphi_{Nj}, \varphi_{Nj}\rangle \to \int_{\mathbb{T}^2} f(\xi) d\xi \tag{65}$$

as $N \to \infty$, for all j in an index set J_N of full density, i.e., $\#J_N \sim N$. Kurlberg and Rudnick [9] have characterized special bases of eigenfunctions $\{\varphi_{Nj}\}_{j=1}^N$ (termed *Hecke eigenbases* in analogy with arithmetic surfaces) for which QUE holds, generalizing earlier work of Degli Esposti, Graffi and Isola (1995). That is, (65) holds for *all* $j = 1, \ldots, N$. Rudnick and Kurlberg, and more recently Gurevich and Hadani, have established results on the rate of convergence analogous to (49). These results are unconditional. Gurevich and Hadani use methods from algebraic geometry based on those developed by Deligne in his proof of the Weil conjectures (an analogue of the Riemann hypothesis analogue for finite fields).

In the case of quantum cat maps, there are values of N for which the number of coinciding eigenvalues can be large, a major difference to what is expected for the modular surface. Linear combinations of eigenstates with the same eigenvalue are as well eigenstates, and may lead to different quantum limits. Indeed Faure, Nonnenmacher and De Bièvre [5] have shown that there are subsequences of values of N, so that, for all $f \in C^\infty(\mathbb{T}^2)$,

$$\langle \mathrm{Op}(f)\varphi_{Nj}, \varphi_{Nj} \rangle \to \frac{1}{2} \int_{\mathbb{T}^2} f(\xi) d\xi + \frac{1}{2} f(0), \tag{66}$$

i.e., half of the mass of the quantum limit localizes on the hyperbolic fixed point of the map. This is the first, and to date only, rigorous result concerning the existence of scarred eigenfunctions in systems with chaotic classical limit.

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Wave Functions For Classically Chaotic Systems

Consider a chaotic billiard in 2-d, e.g. Bunimovich Stadium

Consider w.f. for $\lambda << L$ when L is some characteristic size of the billiard. We are in the semi-classical regime. We discuss this in more detail soon.

Berry made some interesting arguments in 1977

Classical motion on a const. energy surface

$$\rho_N(\vec{r}^N, \vec{p}^N) = \frac{\delta(E - H(r^N, p^N))}{\int d\Gamma\, \delta(E - H(r^N, p^N))}$$

$$= \frac{\delta\left(E - \sum \frac{p_i^2}{2m} - V(r^N)\right)}{\int d\Gamma\, \delta\left(E - \frac{\sum p_i^2}{2m} - V(r^N)\right)}$$

$$f(r^N) = \int d^N p\, \rho_N(r^N, p^N)$$

Now f_N depends only on magnitude of $\vec{P}$ not on direction.

Also the walls produce random trajectories

$$f(\vec{r}) \approx \frac{1}{\text{Area of billiard}} \qquad d=2$$

Berry Conjecture: For short wave lengths $\psi(\vec{r})$ in a chaotic billiard is a random superposition of plane waves with same $|\vec{k}|$

$$\psi(\vec{r}) = \sum_{n}^{N} a_n e^{i\vec{k}_n\cdot\vec{r}} \qquad |\vec{k}_n| = k$$

What does this mean

(1) Normalization

$$1 = \int_A d\vec{r}\, |\psi(\vec{r})|^2 = A\sum |a_n|^2 + \sum_{m\neq n} \underbrace{\int d\vec{r}\, a_n a_m^* e^{i k_n\cdot r} e^{-ik_m\cdot r}}_{\approx 0 \text{ by oscillations}}$$

$$1 \approx A \sum |a_n|^2$$

on average $\langle |a_n|^2 \rangle = \frac{1}{NA}$

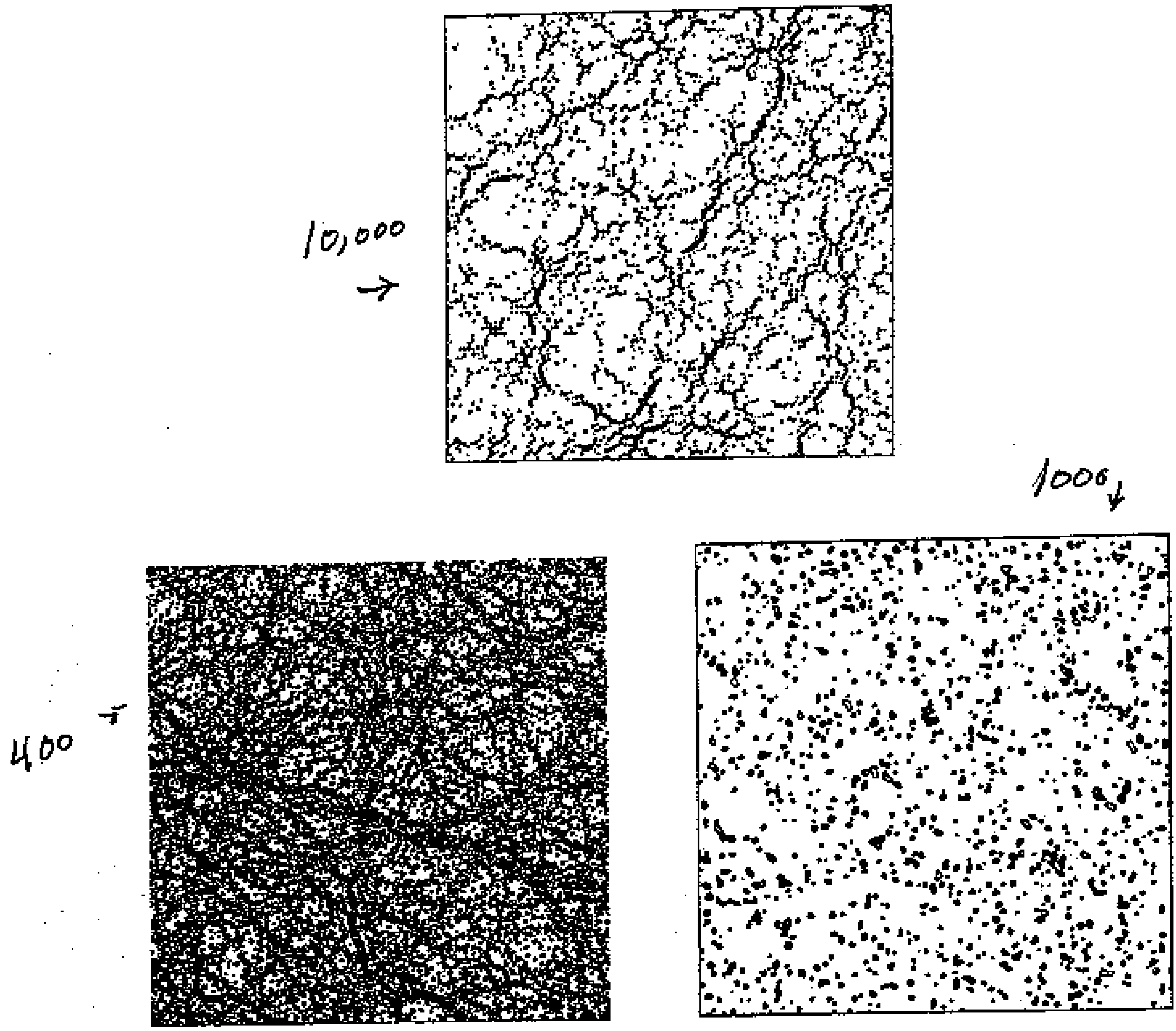


Fig. 42. (Left) A superposition of 10 000 plane waves of random direction, amplitude, and phase shift. The wavevector magnitude was the same for all the plane waves used. The left and middle plots are 60 wavelengths on a side. This state satisfies $\nabla^2\psi + k^2\psi = 0$. (Middle) The probability contours of a wave similar to the one at the left. (Right) A speckle pattern (probability contours), produced in the same way except that various wavevector magnitudes were used. This plot is ca. 20 wavelengths across.

(2) Prob distrib of amplitudes

Suppose ψ is real

$$\psi(r) = \sum a_n \cos(k_n r + \phi_n)$$

$$P(\psi = \Psi) = \left\langle \delta\left(\Psi - \sum a_n \cos(k_n r + \phi_n)\right)\right\rangle$$

averaged over amplitudes, direction + phases of waves

$$= \left\langle \frac{1}{2\pi}\int dz\, e^{iz\left(\Psi - \sum a_n \cos(k_n r + \phi_n)\right)}\right\rangle$$

$$= \frac{1}{2\pi}\int dz\, e^{iz\Psi} \prod_n \left\langle e^{-iza_n \cos(k_n r + \phi_n)}\right\rangle$$

$$\left\langle e^{-i\alpha}\right\rangle = 1 - i\langle\alpha\rangle - \frac{1}{2}\langle\alpha^2\rangle +$$

$$\left\langle e^{-iza_n\cos(k_n r + \phi_n)}\right\rangle = 1 - iz\langle a_n \cos(k_n r + \phi_n$$

$$-\frac{z^2}{2}\left\langle a_n^2 \cos^2(k_n \phi + \phi)\right\rangle = 1 - \frac{z^2}{4AN}$$

$$P(\psi = \Psi) \approx \int_{-\infty}^{\infty} dz\, e^{iz\Psi - \frac{z^2}{A}}$$

CENTRAL LIMIT THM.

Gaussian distrib of sum of random variables

$$\approx e^{-\frac{A\Psi^2}{4}} \quad \text{Gaussian in } \Psi$$

Spatial Correlations

$$\langle \psi(\vec{r}+\vec{R})\, \psi(\vec{r}) \rangle$$

$$= \sum_{n,m} \langle a_n a_m \cancel{\oint} \cos(k_n(r+R) + \phi_n) \cos(k_m r + \phi_m) \rangle$$

$$\frac{1}{2} \{ \cos[r(k_n + k_m) + k_n R + \phi_n + \phi_m]$$

$$+ \cos[r(k_n - k_m) + k_n R + \phi_n - \phi_m] \}$$

Assume $a_n + a_m$ are uncorrelated unless $n = m$

$n = m$

$$\langle \psi(r+R)\psi(r) \rangle = \frac{1}{2} \sum_n \langle a_n^2 [\cos(2k_n r + k_n R + 2\phi_n)$$

$$+ \cos(k_n R)] \rangle$$

averaging kills the first term

$$\langle \psi(r+R)\psi(r) \rangle = \frac{1}{2} \sum_n \langle a_n^2 \cos k_n R \rangle$$

again

$$\langle a_n^2 \rangle = \frac{1}{NA}$$

$$\langle \psi(r+R)\,\psi(r) \rangle = \frac{1}{2A} \langle \cos k_n R \rangle$$

$$\langle \cos k_n R \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos(kR\cos\phi) = J_0(kR)$$

$$\langle \psi(r+R)\,\psi(r) \rangle = \frac{1}{2A} J_0(kR)$$

In d-dimensions

$$\langle \psi(r+R)\,\psi(r) \rangle \sim \frac{J_{\frac{d}{2}-1}(kR)}{(kR)^{\frac{d}{2}-1}}$$

[illegible]

$$P_2(r) = 1 + cJ_0^2(kr), \qquad (2)$$

Sinai Stadium
= Bunimovich Stadium ($\frac{1}{4}$ of it)
with circular scatterer
to remove bouncing orbits.

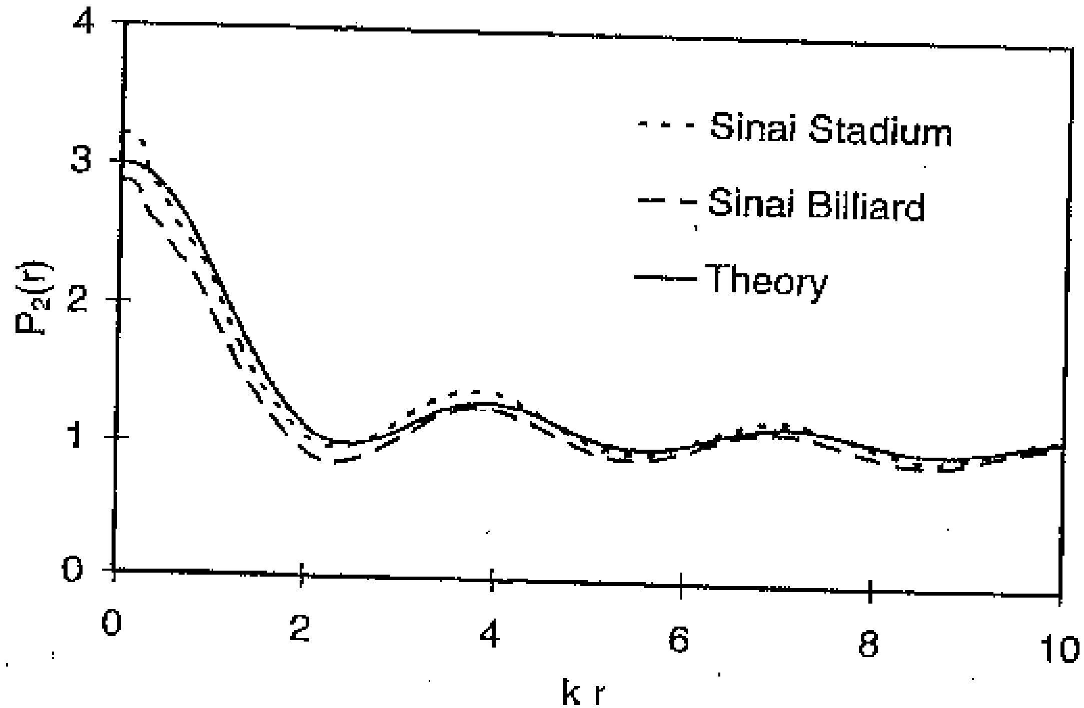


FIG. 3. Density correlation function $P_2(r)$ for the chaotic geometries. The solid line represents Eq. (2) with $c = 2$, as expected from a supersymmetry approach.

Kudrolli, Kidambi & Sridhar

PRL 75, 822 (1995)

Schnirelman's Theorem

We saw earlier that, on average, the small wavelength wave functions are uniform in the Bunimovich Stadium. This is actually an example of Schnirelman's Thm:

Roughly speaking it says that as the wavelength the becomes small for a CCS, the wave functions become uniformly distributed ~~[illegible]~~

Here is a simple version.

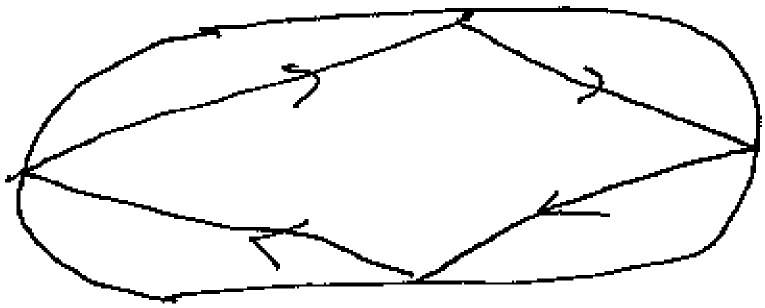
Consider a classically chaotic billiard, B. There is a sequence of integers $(j_k)_{k \in \mathbb{N}}$ such that for every region A of positive measure in B, one has

$$\lim_{k \to \infty} \int_A |\psi_{j_k}(\vec{r})|^2 \, d\vec{r} = \frac{\mu(A)}{\mu(B)}$$

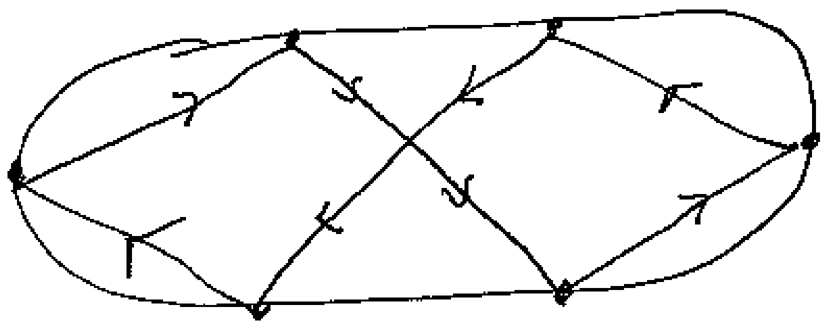
Proof: See de Bièvre: Quantum Chaos, A Brief First Visit

Periodic Orbits & Scarring

Classically, a ccs has a dense set of periodic orbits. We'll look more closely at this later on



For example



Heller discovered that there exist wave functions that are concentrated on periodic orbits. These wave functions are scarred.

HELLER FIGURE 2

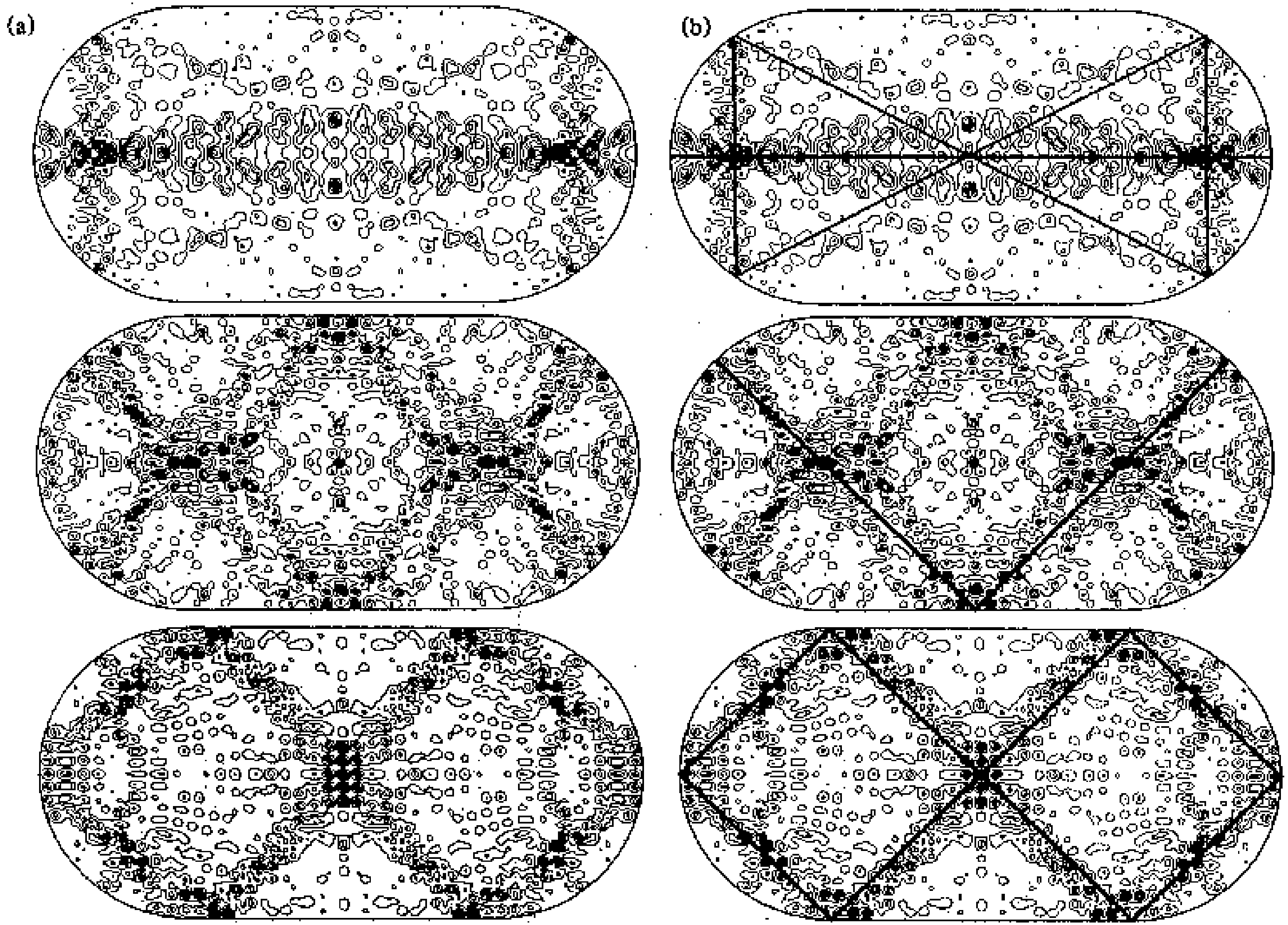


FIG. 2. Left column, three scarred states of the stadium; right column, the isolated, unstable periodic orbits corresponding to the scars.

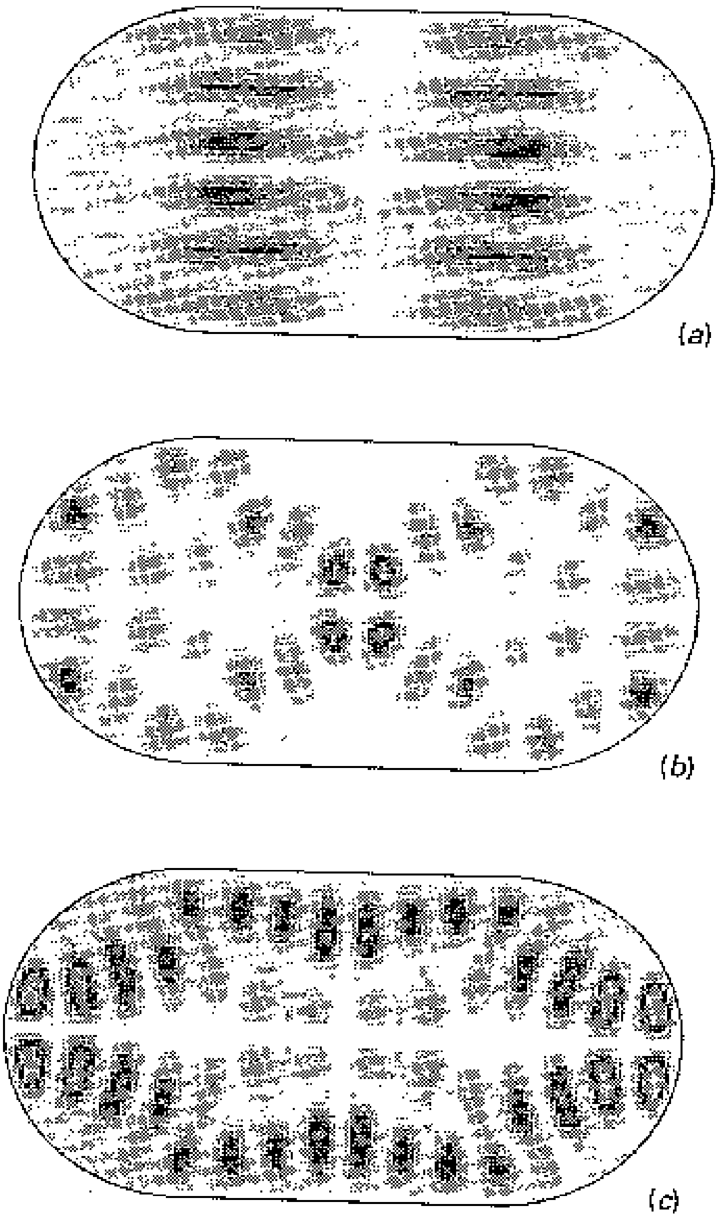


Fig. 9. Experimentally determined eigenfunctions for a stadium billiard [11]. The eigenfunctions show scars corresponding to the bouncing ball (a), the double diamond (b), and the whispering gallery orbit (c).

Do Scars contradict Shnirelman's Thm?

No: Two reasons.

1) S thm is about integrals of the w.f. + Scars are concentrated in very small regions

2) proportion of scarred eigenfns goes to 0 as $f_k \to \infty$

See O'Connor + Heller PRL 61, 2288 (1988).

To go further + to do much more on this.

We need to consider now the semiclassical limit!

SEMI-CLASSICAL METHODS

(1) WKB-method, integrable systems

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\psi(r,t) = A(r,t)\, e^{\frac{iR(\vec{r},t)}{\hbar}} \qquad A, R \text{ real}$$

Hamilton JACOBI EQ

$$-\frac{\partial R}{\partial t} = \frac{(\nabla R)^2}{2m} + V(r) \cancel{-\frac{\hbar^2}{2m}\frac{\nabla^2|A|}{|A|}}$$

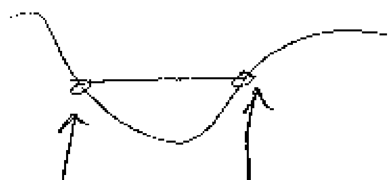
$$\rho = |A|^2$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\,\frac{\nabla R}{m}\right) = 0$$

Time independent form $R = S - Et$

$$E - V = \frac{(\nabla S)^2}{2m}$$

$$S = \pm\int_{x_1}^{x} dx'\,[2m(E-V)]^{1/2}$$



1/4 for each reflection

classical turning points

Usual solution has bound state if

$$\oint P\,dx = (n + 1/2)\,h \qquad \text{Bohr-Sommerfeld}$$

If hard wall on one side   wall   change in phase by $\pi/2$ at hard wall

$$\oint p\,dx = \left(n + \frac{1}{2} + \frac{1}{4}\right) h$$

one reflection

Good Survey of WKB in

SEMICLASSICAL PHYSICS, M. Brack + R. Bhaduri

For chaotic systems need to avoid B-S quantization

Feynmann Path Integrals Work Well

$$\langle \vec{r}_B | e^{-\frac{i}{\hbar} t H} | \vec{r}_A \rangle = \int_{\text{Path}} D(\text{path})\, e^{\frac{i}{\hbar}\int_0^t L(\vec{r},\dot{\vec{r}})dt}$$

Integral over all paths from $\vec{r}_A \to \vec{r}_B$, L is the Lagrangian that take place in time t $T - V$

Essential ingredient is the short time propagator $t \to \delta t$

$$\langle \vec{r}_B | e^{-\frac{i}{\hbar}(T+V)\delta t} | \vec{r}_A \rangle$$

$$\langle r_B | e^{-\frac{i}{\hbar}(T+V)\delta t} | \vec{r}_A \rangle \approx \langle r_B | e^{-\frac{i}{\hbar} T \delta t}\, e^{-\frac{i}{\hbar} V \delta t} | \vec{r}_A \rangle$$

$$= \langle r_B | e^{-\frac{i}{\hbar} T \delta t} | r_A \rangle e^{-\frac{i}{\hbar} V(\vec{r}_A)\delta t}$$

$$\langle r_B | e^{-\frac{i}{\hbar} T \delta t} | r_A \rangle = \frac{1}{(2\pi\hbar)^d} \int d\vec{p}\, \langle r_B | p \rangle \langle p | r_A \rangle\, e^{-\frac{i}{\hbar}\delta t \frac{p^2}{2m}}$$

(71)

$$\langle \vec r_B | e^{-\frac{i}{\hbar}\delta t\, T} | \vec r_A \rangle = \frac{1}{(2\pi\hbar)^d} \int d\vec p\; e^{-\frac{i}{\hbar}\delta t \frac{p^2}{2m}}\, e^{i\vec p\cdot(\vec r_A - \vec r_B)}$$

$$= \left(\frac{m}{2\pi i\hbar\,\delta t}\right)^{d/2} e^{\frac{i}{\hbar}\frac{m}{2\delta t}(\vec r_A - \vec r_B)^2}$$

$$\langle \vec r_B | e^{-\frac{i}{\hbar}\delta t\, H} | \vec r_A \rangle = \left(\frac{m}{2\pi i\hbar\,\delta t}\right)^{d/2} e^{\frac{i}{\hbar}\left[\frac{m}{2\delta\tau}(\vec r_A - \vec r_B)^2 - \delta\tau\, V\left(\frac{\vec r_A + \vec r_B}{2}\right)\right]}$$

$$\tilde r_A = \frac{\vec r_A + \vec r_B}{2}$$

$$\frac{(\vec r_A - \vec r_B)^2}{\delta\tau} = \left[\frac{\vec r_A - \vec r_B}{\delta\tau}\right]^2 \delta\tau = v^2\,\delta\tau$$

$$\langle \vec r_B | e^{-\frac{i}{\hbar}\delta t\, H} | \vec r_A \rangle = \left(\frac{m}{2\pi i\hbar\,\delta t}\right)^{d/2} e^{\frac{i}{\hbar}\delta t\left[\frac{m}{2}v^2 - V\right]}$$

Sloppy Trick

Let

$$W_{BA} = \frac{m}{2\delta\tau}(r_A - r_B)^2 - \delta\tau\, V(\tilde r_A)$$

For small $\delta\tau$

$$\frac{m}{\delta\tau} = -\frac{\partial^2 W_{BA}}{\partial r_{A_i}\partial r_{B_j}}, \quad \left(\frac{m}{\delta\tau}\right)^d = \mathrm{Det}\left[-\frac{\partial^2 W_{AB}}{\partial \vec r_A\,\partial \vec r_B}\right]$$

So

$$\langle \vec r_B | e^{-\frac{i}{\hbar}\delta t H} | r_A \rangle = \left(\frac{1}{2\pi i\hbar}\right)^{d/2} \left| -\frac{\partial W_{BA}}{\partial \vec r_A \partial r_B} \right|^{1/2} e^{\frac{i}{\hbar}\underbrace{\int_0^{\delta t} L\, dt}_{W_{BA}}}$$

Now we have to put these things together to get a finite time propagator.

$$\langle \vec r_B | e^{-\frac{i}{\hbar} t H} | r_A \rangle$$

$$= \int d\vec r_1 \cdots \int d\vec r_N \; \langle r_B | e^{-\frac{i}{\hbar}\delta t H} | r_1 \rangle \langle r_1 | e^{-\frac{i}{\hbar}\delta t H} | r_2 \rangle \langle r_2 | e^{-\frac{i}{\hbar}\delta t H} | r_3 \rangle \cdots \langle \vec r_N | e^{-\frac{i}{\hbar}\delta t H} | r_A \rangle$$

We'll use stationary phase approximation

Consider

$$\langle r_B | e^{-\frac{i}{\hbar} 2\delta t H} | r_A \rangle$$

$$= \int d\vec r_1 \, \langle r_B | e^{-\frac{iH}{\hbar}\delta t} | \vec r_1 \rangle \langle r_1 | e^{-\frac{iH}{\hbar}\delta t} | \vec r_A \rangle$$

$$\langle \vec{r}_B | e^{-\frac{i}{\hbar} 2\delta\tau H} | \vec{r}_A \rangle$$

$$= \left(\frac{1}{2\pi i\hbar}\right)^{\frac{d}{2}\cdot 2} \int d\vec{r}_1 \left| -\frac{\partial W_{B1}}{\partial \vec{r}_B \partial \vec{r}_1} \right|^{1/2} \left| \frac{-\partial W_{1A}}{\partial \vec{r}_1 \partial r_A} \right| e^{\frac{i}{\hbar}[W_{B1} + W_{1A}]}$$

Phase $\quad W_{B1} + W_{1A} = \Phi(\vec{r}_1)$

expand $\Phi(\vec{r}_1)$ about a stationary point $\vec{r}_1^{\,o}$

$$\Phi(\vec{r}_1) = \Phi(\vec{r}_1^{\,o}) + (\vec{r}_1 - \vec{r}_1^{\,o}) \cdot \frac{\partial \Phi(\vec{r}_1^{\,o})}{\partial r_1^o} + \frac{1}{2}(\vec{r}_1 - \vec{r}_1^{\,o})^2 : \frac{\partial^2 \Phi}{\partial r_1^{o\,2}}$$

We need

$$\frac{\partial \Phi(r_1^o)}{\partial r_1^o} = 0 = \frac{\partial}{\partial r_1^o}\left[W_{Br_1^o} + W_{r_1^o A}\right]$$

$$= \vec{V}_{1^o \to B} + \vec{V}_{A \to 1^o} = 0$$

Velocity is continuous at stationary point

$$\langle \vec{r}_A | e^{-\frac{i}{\hbar} S\tau H} | \vec{r}_B \rangle = \left(\frac{1}{2\pi\hbar i}\right)^{\frac{d}{2}\cdot 2} e^{\frac{i}{\hbar}[W_{B1^o} + W_{1^o A}]}$$

$$\int d\vec{r}_1 \; e^{\frac{i}{\hbar}(\vec{r}_1 - r_1^o)^2 : \frac{\partial^2 \Phi}{\partial r_1^o}} \left| D_{B1} D_{1A} \right|^{1/2}$$

$\Phi = W_{B1} + W_{1A}$

Integral (when mode finite)

$$\sim \frac{1}{\left[\mathrm{Det} \left[\frac{\partial^2 W_{B1}}{\partial r_1^2} + \frac{\partial^2 W_{1A}}{\partial r_1^2} \right]_{r_1 = r_1^0} \right]^{1/2}}$$

We have
$$\left[\frac{\mathrm{Det} \left| -\frac{\partial^2 W_{B1}}{\partial r_B \partial r_1} \right| \mathrm{Det} \left| -\frac{\partial^2 W_{1A}}{\partial r_1 \partial r_A} \right|}{\mathrm{Det} \left[\frac{\partial^2 W_{B1}}{\partial r_1^2} + \frac{\partial^2 W_{1A}}{\partial r_1^2} \right]} \right]^{1/2}_{r_1 = r_1^0}$$

After a lot of manipulation we get

$$\langle r_B | e^{-\frac{i}{\hbar} 2\delta t H} | r_A \rangle = \left(\frac{1}{2\pi i \hbar} \right)^{d/2} \left[\mathrm{Det} \left(\frac{\partial W_{BA}}{\partial r_A \partial r_B} \right) \right]^{1/2}$$

$$e^{\frac{i}{\hbar} W_{BA}(2\tau)}$$

$W_{BA}(2\tau) = \int_0^{2\tau} dt\, L$ classical path from $\vec{r}_A$ to $\vec{r}_B$ in $2\delta\tau$

The full propagator would then seem to be

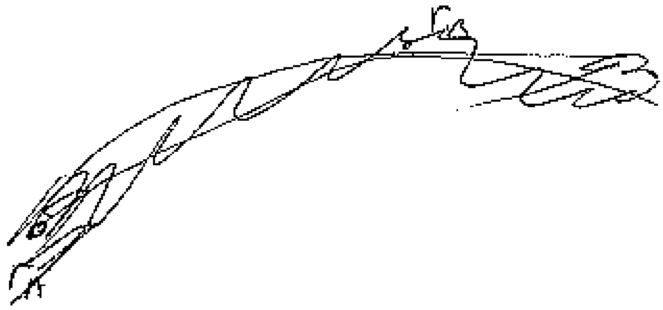
$$K(r_B, r_A, t) = \langle \vec{r}_B | e^{-\frac{i}{\hbar} t H} | \vec{r}_A \rangle$$

$$\approx \left(\frac{1}{2\pi i \hbar}\right)^{d/2} \sum_{\text{paths}} \left[\mathrm{Det} \left| \frac{-\partial^2 W_{BA}}{\partial r_A \partial r_B} \right| \right]^{1/2} e^{\frac{i}{\hbar} W_{BA}(t)}$$

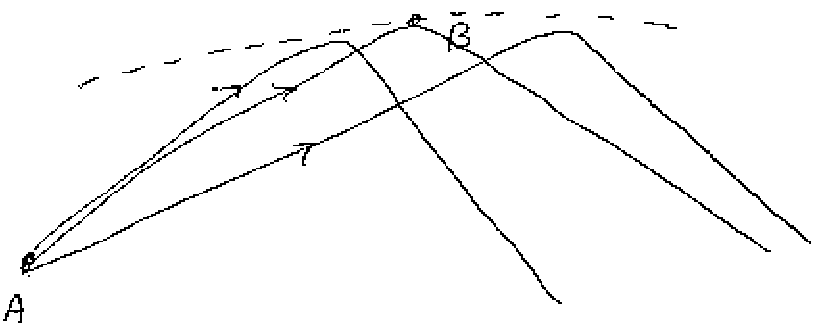
$$W_{BA}(t) = \int_0^t d\tau \; L(\tau)$$ action of a classical path

However there is still an issue of phase factor. There is a problem whenever $\frac{\partial^2 W_{BA}}{\partial r_A \partial r_B}$ becomes singular $\sim \left|\frac{\partial p_A}{\partial r_B}\right|$ become singular.

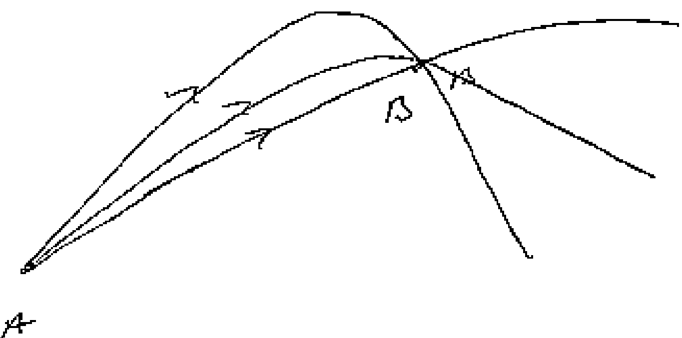
This happens whenever $\left|\frac{\partial r_B}{\partial p_A}\right| = 0$ can change p_A without changing r_B. At a <u>caustic</u> can change



$\vec{p}_A$ without changing at least one component of r_B



Or at a focal point



Then

$$\langle r_B | e^{-\frac{i}{\hbar} t H} | r_A \rangle = \left(\frac{1}{2\pi i \hbar}\right)^{d/2} \sum |D_{BA}|^{1/2} e^{\left(\frac{i}{\hbar} W_{BA}(t) - \frac{i \nu_p \pi}{2}\right)}$$

$\nu =$ ~~# of negative eigenvalues of~~ number of conjugate points along path P, with possible multiplicities

go through the momentum representation to get the phase factor

Propagator PATH INTEGRALS, L. Shulman, Wiley

$$\langle \vec{r}_B | e^{-\frac{i}{\hbar} t H} | r_A \rangle = \left(\frac{1}{2\pi i \hbar}\right)^{d/2} \sum_{\text{paths}} |D_{BA}|^{1/2} e^{\left[\frac{i}{\hbar}\int_0^t dz\, L(z) - \frac{i \nu_p \pi}{2}\right]}$$

We want to get to the Gutzwiller Trace Formula For the Density of States $\rho(E)$

$$\rho(E) = \sum_{\text{states } n} \delta(E - E_n)$$

Use identity $\delta(E) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \frac{1}{E^2 + \varepsilon^2}$

$$= -\lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \frac{1}{E + i\varepsilon}$$

$$\rho(E) = -\lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \sum_n \frac{1}{(E - E_n) + i\varepsilon}$$

$$= -\lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \frac{1}{E - H}$$

$$= -\lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \int d\vec{r}\, \underbrace{\langle \vec{r} | \frac{1}{E - H} | \vec{r} \rangle}_{\text{Green's Function}}$$

28

$$\langle \vec{r}_B | \frac{1}{E-H} | \vec{r}_A \rangle = G(\vec{r}_B, \vec{r}_A, E)$$

$$= -\frac{i}{\hbar} \int_0^\infty d\tau\, e^{\frac{i\tau E}{\hbar}} \langle r_B | e^{-\frac{i\tau}{\hbar} H} | r_A \rangle$$

with a convergence factor implied

= Laplace Transform of Propagator

$$= -\frac{i}{\hbar} \left(\frac{1}{2\pi i \hbar} \right)^{d/2} \sum_{\text{paths } p} \int_0^\infty dt\, |D_{BA,p}|^{1/2} e^{\frac{i}{\hbar}\left[tE + \frac{i}{\hbar}\int_0^t L\,dt - \frac{i\pi}{2}\nu_p \right]}$$

Stationary phase trick

$$e^{\frac{i}{\hbar}\left[tE + W_{AB}(t) \right]}$$

$$e^{\frac{i}{\hbar}\left[\tau_0 E + W_{AB}(\tau_0) + (t-\tau_0)\left[E + \frac{\partial W_{BA}}{\partial t}\Big|_{\tau_0} \right] \right.}$$

$$e^{\frac{i}{2\hbar}(t-\tau_0)^2 \frac{\partial^2 W_{AB}}{\partial t^2}\Big|_{\tau_0} + \cdots}$$

e

$$E + \left.\frac{\partial W_{BA}}{\partial t}\right|_{\tau_0} = 0$$ determines τ_0 for each path

Hamilton-Jacobi equation

$$t_0 E + W_{BA}(t_0) = S(\vec{r}_B, \vec{r}_A, E)$$ action

Digression

$$W_{BA} = \int_0^t d\tau\, L(\tau)$$

$$L(\tau) = \sum p \dot{q}_i - H$$

$$W_{BA} = \int_0^t d\tau \left(\sum p \dot{q}_i - H \right)$$

$$= \sum \int p_i dq_i - Et$$

$$\sum \int p_i dq_i = W_{BA} + Et$$ Action S

$$G(\vec{r}_B, \vec{r}_A, E) = -\frac{i}{\hbar}\left(\frac{1}{2\pi i\hbar}\right)^{d/2} \sum_{\substack{\text{paths} \\ p}} |D_{BA,p}(t_0)|^{1/2}$$

$$e^{\frac{i}{\hbar} S(\vec{r}_B, \vec{r}_A, \vec{E})} \int_0^\infty dt\, e^{\frac{i(t-t_0)^2}{2} \frac{\partial^2 W}{\partial t^2}}$$

$$\int_0^\infty e^{i\alpha t^2 - \varepsilon t^2} = \int_0^\infty dt\, e^{-t^2(\varepsilon - i\alpha)}$$

$$e^{i\frac{3\pi}{4} 2}$$

$$i\alpha t^2 - \varepsilon t^2$$

$$\approx \frac{1}{\sqrt{\varepsilon - i\alpha}} \rightarrow \frac{1}{(e^{\frac{3i\pi}{4}})\alpha^{1/2}}$$

$$G(\vec{r}_B, \vec{r}_A, E) = -\frac{1}{\hbar} \frac{e^{-\frac{3\pi i}{4}}}{(2\pi i\hbar)^{d/2}} \sum \left[\frac{D_{BA,p}}{\frac{\partial^2 W}{\partial t^2}\big|_{t_0}}\right]^{1/2}$$

$$e^{\frac{i}{\hbar} S(r_B, r_A, E) - i\frac{\nu\pi}{2}}$$

Now there is a lot of manipulations with the coefficient which I skip (Stöckmann, etc.)

Leading to

$$G(\vec{r}_B, \vec{r}_A, E) = \frac{-i}{\hbar}\left(\frac{1}{2\pi i \hbar}\right)^{\frac{d-1}{2}} \sum_{\text{paths}} |\Delta_{BA,p}|^{1/2}\, e^{\frac{i}{\hbar} S_p(\vec{r}_B, \vec{r}_A, E) - \nu_p \frac{i\pi}{2}}$$

$$|\Delta_{BA}| = \operatorname{Det}\begin{bmatrix} -\dfrac{\partial^2 S}{\partial r_A \partial r_B} & -\dfrac{\partial^2 S}{\partial r_A \partial E} \\[2ex] -\dfrac{\partial^2 S}{\partial r_A \partial E} & -\dfrac{\partial^2 S}{\partial E^2} \end{bmatrix}$$

$$= \begin{bmatrix} \overset{\frac{\partial^2 S}{\partial r_A \partial r_B}}{\begin{bmatrix} 3\times 3 \end{bmatrix}} & \overset{\frac{\partial^2 S}{\partial r_A \partial E}}{\begin{bmatrix} 3\times 1 \end{bmatrix}} \\[2ex] [1\times 3] & [1\times 1] \\ \begin{bmatrix} 3\times 1 \end{bmatrix} \dfrac{\partial^2 S}{\partial E \partial r_A} & \dfrac{\partial^2 S}{\partial E^2} \end{bmatrix}$$

For billiards

$$S = \int p\,dq = p\,l$$

p = momentum, l = length of path

Now we can continue

$$\rho(E) = -\frac{1}{\pi}\,\mathrm{Im}\int d\vec{r}\; G(\vec{r},\vec{r},E)$$

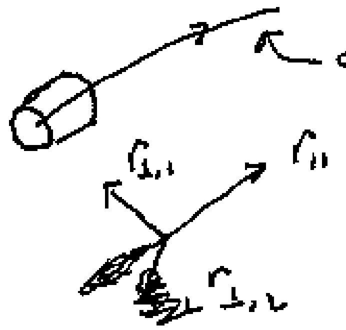
$$\int d\vec{r}\; G(\vec{r},\vec{r},E) = -\frac{i}{\hbar}\left(\frac{1}{2\pi i\hbar}\right)^{\frac{d-1}{2}}$$

$$\sum_p \int d\vec{r}\, |\Delta_p|^{1/2}\, e^{\left[\frac{i}{\hbar} S(\vec{r},\vec{r},E) - \frac{i\nu\pi}{2}\right]}$$

Sum is over all closed paths starting & ending at $\vec{r}$.

What about paths of zero length which maybe some singular limit?

We'll use stationary phase approx again



$$d\vec{r} = dr_{\parallel} \prod_j dr_{\perp,j}$$

$$S(\vec{r}, \vec{r}, E) = S(r_{\parallel}, \vec{r}_{\perp}; r_{\parallel}, \hat{r}_{\perp}, E)$$

We suppose that we can expand S in powers of $r_\perp$

$$S(\vec{r}, \vec{r}, E) = S(r_{\parallel}, r_{\parallel}, E) + r_\perp \frac{\partial S(\vec{r}, \vec{r}, E)}{\partial r_\perp}$$

$$+ \frac{\partial S}{\partial r_\perp} r_\perp + \frac{1}{2} r_\perp \frac{\partial^2 S}{\partial r_\perp \partial r_\perp} r_\perp + \cdots$$

$$S(r_{\parallel}, r_{\parallel}, E) = \oint p_{\parallel} \, dq_{\parallel} \quad \text{around an orbit}$$

the linear term vanishes

$$\left. r_{B\perp} \frac{\partial S}{\partial r_{B\perp}} + \frac{\partial S}{\partial r_{A\perp}} r_{A\perp} \right|_{\vec{r}_A = \vec{r}_B} = r_\perp (p_\perp - p_\perp) = 0$$

So we consider the derivative of S in the $\perp$ plane

The $\perp$ plane is governed by the Monodromy Matrix M which relates ~~the~~ the $\perp$ components at r_B to those at $\vec{r}_A$

$$\begin{pmatrix} \delta r_{B\perp} \\ \delta p_{B\perp} \end{pmatrix} = M_{BA} \begin{pmatrix} \delta r_{A\perp} \\ \delta p_{B\perp} \end{pmatrix}$$

For a periodic orbit A + B are the same + M describes the growth of an infinitesimal perturbation about a closed orbit

In terms of M one derives eventually

$$\int d\vec{r}\, G(\vec{r},\vec{r},E) = -\frac{i}{\hbar} \sum_{cp} \underbrace{\frac{\overbrace{(T_p)_r}^{\text{period of the orbit}}}{\| M_p - 1 \|^{1/2}}}_{\text{determinant}} e^{i\left(\frac{1}{\hbar} S_p - \frac{\pi \overbrace{\nu_p}^{\text{Maslov index}}}{2}\right)}$$

INCLUDES REPETITIONS

T_p is period of primitive orbit

$$\rho(E) = \frac{1}{\pi\hbar} \sum \frac{(T_p)_p}{\| M - 1 \|^{1/2}} \cos\left[\frac{S}{\hbar} - \frac{\pi}{2}\nu\right]$$

There is another contribution coming from singular orbits of zero length: This is the usual density of states classically

$$\rho_0(E) = \int \frac{d\vec{r}\,d\vec{p}}{(2\pi\hbar)^d}\,\delta(H(\vec{r},\vec{p}) - E)$$

$$\simeq V(\quad)\,E^{\frac{d-2}{2}}$$

$$\rho_{tot}(E) = \rho_0(E) + \rho_{GTF}(E)$$

SEE ALSO

B. Mehlig + M. Wilkenson

"Semi-Classical Trace Formulae Using Coherent States"

Ann. Phys. (Lpz.) 10, 541 (2001), Cond-mat 001202

Applications of Gutzwillers Trace Formula

Verification of RMT for smooth hyberbolic Systems via "Sieber-Richter Pairs".

Consider Spectral Density function

$$\langle \rho(E+\tfrac{\varepsilon}{2})\,\rho(E-\tfrac{\varepsilon}{2})\rangle_E$$ average over many states close to E - but interval small w.r.t. E

If we write $\rho(E) = \rho_0(E) + \rho_{GTF}(E)$

Then only terms that contribute are $\langle \rho_0 \rho_0\rangle + \langle \rho_G \rho_G\rangle$

The interesting part is $\langle \rho_G \rho_G\rangle$

Spectral Form Factor $K(\tau)$

$$K(\tau) = \int_{-\infty}^{\infty} \frac{d\eta}{\rho_0(E)} \left\langle \rho_G(E+\tfrac{\eta}{2})\,\rho_G(E-\tfrac{\eta}{2})\right\rangle_E e^{2\pi i \eta \tau \rho_0(E)}$$

For RMT + GOE

$$K^{GOE}(\tau) = \begin{cases} 2\tau - \tau \log(1+2\tau) & \tau < 1 \\ 2\tau - \tau \log \dfrac{2\tau+1}{2\tau-1} & \tau > 1 \end{cases}$$

The units are in terms of the Heisenberg Time

$T_H = 2\pi\hbar\bar{\rho}(E)$ $\quad \left[\tau = \dfrac{t}{\hbar\rho_0(E)}\right]$

Roughly the time needed to resolve level spacings

$$\rho_G(E) = \sum_\gamma \hat{A}_\gamma \cos\left(\frac{S_\gamma(E)}{\hbar} - \frac{\pi}{2}\nu_\gamma\right)$$

$$= \mathrm{Re} \sum_\gamma A_\gamma e^{\frac{i S_\gamma(E)}{\hbar}}$$

stick this low here

$$\left\langle \rho_G\left(E+\frac{1}{2}\right)\rho_G\left(E-\frac{1}{2}\right)\right\rangle = \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'} e^{\frac{i}{\hbar}\left(S_\gamma(E+\frac{1}{2}) - S_{\gamma'}(E-\frac{1}{2})\right)}$$

$$S_\gamma\left(E \pm \frac{1}{2}\right) = S_\gamma(E) \pm \frac{1}{2}\frac{\partial S_\gamma(E)}{\partial E}$$

T_γ

$(\delta\gamma')$

$$\langle \rho_G(E+\tfrac{\eta}{2})\,\rho_G(E-\tfrac{\eta}{2})\rangle = \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'}\, e^{\frac{i}{\hbar}(S_\gamma(E) - S_{\gamma'}(E))}$$

$$e^{\frac{i\eta}{2}(T_\gamma + T_{\gamma'})}$$

η integral gives

$$\delta\left(\tau T_H - \left(\frac{T_\gamma + T_{\gamma'}}{2}\right)\right)$$

$$K(\tau) = \text{Const} \cdot \sum_{\gamma,\gamma'} \langle A_\gamma A^*_{\gamma'}\, e^{\frac{i}{\hbar}(S_\gamma(E) - S_{\gamma'}(E))}\, \delta\left(\tau T_H - \frac{T_\gamma + T_{\gamma'}}{2}\right)$$

Diagonal Approx $\gamma = \gamma'$

$$K_0(\tau) = \text{const} \left\langle \sum_\gamma |A_\gamma|^2\, \delta(\tau T_H - T_\gamma) \right\rangle_E$$

Actually if the system is time reversible the time reversed trajectory has the same action

$$S = \int p\, dq \qquad \begin{matrix} (p \to -p) \\ dq \to -dq \end{matrix}$$

So we can pair up trajectories & their time reversed conjugate

Then $K_0(\tau) = g\,(\text{const}) \left\langle \sum |A_\gamma|^2 \delta(\tau T_H - T_\gamma) \right\rangle_E$

$g = 1$ no time reversal

$\quad 2$ time reversal symmetry

No there is a sum-rule due to Hannay & Ozorio de Almeida

$$|A_\gamma|^2 \approx \frac{T_\gamma^2}{\text{Det}\,\|M_\gamma - \mathbb{I}\|}$$

NEED EQUALITY OF TOPOLOGICAL + METRIC ENTROPIES

$\dfrac{1}{\text{Det}\,\|M_\gamma - \mathbb{I}\|}$ decreases exponentially but the number of periodic orbits increase exponentially for a chaotic system.

As a result (Sum rule of HOdA)

$$\left\langle \sum |A_\gamma|^2 \delta(\tau T_H - T_\gamma) \right\rangle \approx \tau$$

— chaotic

Const – non chaotic integrable

and when all constants are included

$$K_0(\tau) = g\tau$$

~~[illegible]~~

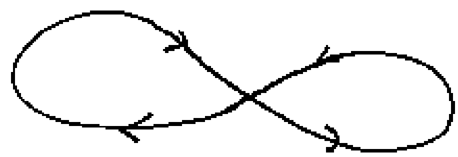
$\tau < 1$ but still long

With HOdA sum rule + $\gamma = \gamma'$ no time reversed terms, we get leading term in form factor

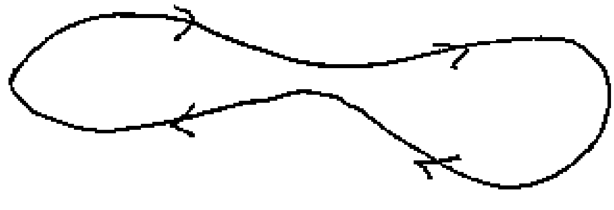
To do better is difficult + solved by M. Sieber + K. Richter. by introducing Sieber-Richter pairs

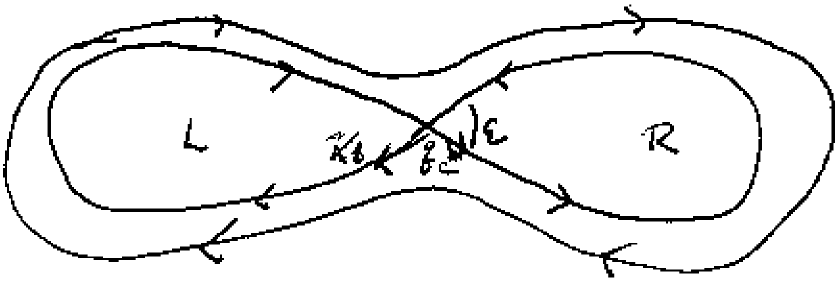
Two orbits with actions that are very close but not identical.

One of the pair has a self-crossing in config space



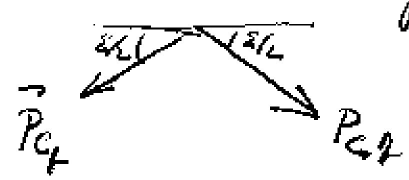
Other is very close to it but no crossing



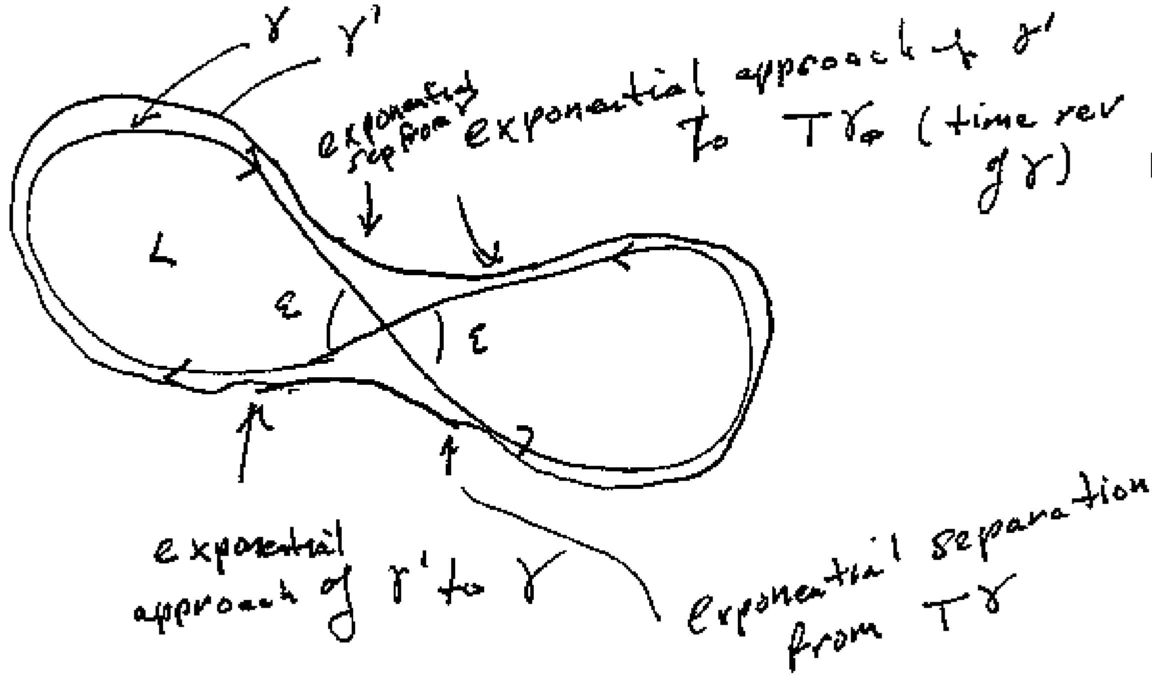


In L orbits are in same direction

In R opposite direction - doesn't affect the action



$|P_{c1} + P_{c2}| \approx P_c \sin \varepsilon$
$\approx P_c \varepsilon$ small ε



S+R proved that ① γ' exists for small ε

② $\gamma + \gamma'$ have almost same period & almost same monodromy matrix

so $A_\gamma = A_{\gamma'}$

③ $$\delta S = S_\gamma - S_{\gamma'} \approx \frac{E\varepsilon^2}{\lambda}$$

$E = \frac{p^2}{2m}$, λ = Lyapunov exponent

Aleiner, Larkin, Sieber, Richter, Agam, ...

Then

$$K^{(2)}(\tau) \simeq \int_{-\pi}^{\pi} d\varepsilon \sum_{\gamma} A_\gamma^2 \, P(\varepsilon, T)\, e^{\frac{i}{\hbar}\delta S}\, \delta(T - T_\gamma)$$

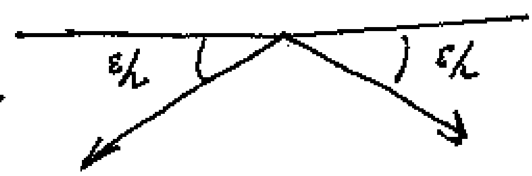
$P(\varepsilon, T)$ = density of crossings with angle ε for trajectories that take time T.

Crossing occurs if there exist a time t' such that (t is some time on trajectory, t' = crossing time)

$$\vec{r}(t + t') = \vec{r}(t') \quad \text{and}$$

$$\phi(t + t') = \phi(t') + \varepsilon - \pi$$

$\phi_2 = \pi + \frac{\varepsilon}{2}$



$\phi_1 = 2\pi - \varepsilon/2$

$$\boxed{\phi_2 = \phi_1 - \pi + \varepsilon}$$

$$\delta S = \frac{p^2 \varepsilon^2}{2m\lambda}$$

gives 0   gives RMT

$$P(\varepsilon, T) \simeq \frac{T^2 \langle v^2 \rangle \sin\varepsilon}{2\pi A}\left(1 + \frac{4 \ln \varepsilon}{\lambda}\right)$$

The density of periodic orbits grows exponentially with time of the orbit

$$\rho(T) \approx \frac{e^{\lambda T}}{T} \quad \text{density of orbits}$$

$$|A_\gamma|^2 \approx e^{-\lambda T}$$

$$K^{(2)} \approx (\,)\int d\varepsilon \, \mathrm{Re}\int_0^\infty dT' \, \rho(T') \frac{T^2}{e^{\lambda T'}} \, P(\varepsilon, T) \, e^{\frac{i}{\hbar}\delta S} \, \delta(T-T')$$

$$\approx (\,) \, \mathrm{Re}\int d\varepsilon \, \frac{v^2 T^3 \varepsilon}{\pi A} \, e^{\frac{i p^2 \varepsilon^2}{2m\lambda\hbar}} \, \frac{\log \varepsilon}{\lambda T}$$

Lyap does not appear in final answer

$$= -2\tau^2 \quad \underline{\text{OK}}$$

S + R

Can now do all orders in τ for simple hyperbolic systems + $\tau > 1$ as well. See Haake

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Leading off-diagonal approximation for the spectral form factor for uniformly hyperbolic systems

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Abstract

We consider the semiclassical approximation to the spectral form factor $K(\tau)$ for two-dimensional uniformly hyperbolic systems, and derive the first off-diagonal correction for small τ. The result agrees with the τ^2-term of the form factor for the GOE random matrix ensemble.

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03.65.Sq Semiclassical theories and applications.
05.45.Mt Semiclassical chaos ("quantum chaos").

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