An explicit construction of Parisi landscapes in finite dimension

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Disordered systems and landscapes

- Phenomenological description of glassy systems (glasses, proteins, spin-glasses): rugged energy landscapes

- Long (Goldstein 1969), useful but somewhat misleading tradition – coherence length scale implied

- Classification of random landscapes (lessons from Spin-Glasses)
  - SK model and Parisi's full-RSB: complex, hierarchical landcapes – valleys within valleys, ultrametricity
  - REM and 1-step RSB: random (golf course) landscape on the hypercube

Disordered systems and landscapes

- More generally: models of random fields (turbulence, finance)

Extreme value statistics

- Low temperature physics of disordered systems: statistics of deep energies

- For $M \gg 1$ Gaussian variables, $\epsilon_{\text{max}}(M) = \sigma\sqrt{2\ln M}\left[1 + \frac{u}{2\ln M}\right]$

- Extreme value distributions for IID
  - Exponential variables: Gumbel $-\ G(u) = \exp[-u - \exp[-u]]$
  - Bounded variables: Weibull $-\ H(u) = \mu u^{\mu-1}\exp[u^{\mu}]$, $u > 0$
  - Power-law variables: Fréchet $-\ F(u) = \mu\exp[-u^{-\mu}]/u^{1+\mu}$

Extreme value statistics

- The Random Energy Model:
  - $M = 2^N$, $\sigma = \sqrt{N} \rightarrow \epsilon_{\mathrm{max}} \sim N$ and $O(1)$ energy gap: non trivial thermodynamics
  - Gumbel statistics for low energy states equivalent to (1-Step) RSB and localisation

- Applications: Decaying Burgers' Turbulence with random initial conditions: picks up extremes of the initial field

- FRG for pinned manifolds; Shocks, Cusps and RSB [cf. Balents, JPB, Mézard; Le Doussal, Wiese]: bare disorder evolves with scales following a (functional) Burgers equation – shocks are associated to metastable state formation

Random potential (finite dimension, short range)

- One particle in a short-range correlated Gaussian random potential $V(\mathbf{r})$ in N dimensions:

$$V_{\text{max}}(R) \sim \sigma\sqrt{\ln R^N}$$

- Not strong enough to compete with entropy: $S \sim \ln R^N \rightarrow$ always in the high temperature (delocalized) phase

- Diffusive motion at any T, in any N: $D \sim \exp(-\sigma^2/2NT^2)$

- But if the potential has exponential tails, $V_{\text{max}} \sim \sigma N \ln R$: true phase transition of the REM type at $T_c = \sigma$ + aging dynamics below T_c (see e.g ([Ben Arous-Cerny])

Random potential (finite dimension, long range)

- One particle in a long-range correlated Gaussian random potential $V(\mathbf{r})$ in N dimensions:

  $\langle V(\mathbf{r})V(\mathbf{r}')\rangle = Ng^2|\mathbf{r}-\mathbf{r}'|^{2H}, \quad (H > 0) \quad \rightarrow V_{\text{max}}(R) \sim \sqrt{N} g R^H$

- Always beats entropy $S \sim \ln R^N \rightarrow$ always in the low temperature (localized) phase

- Example: Exactly soluble Sinai (random force) model in $N = 1$ dimension $\rightarrow$ logarithmic diffusion and Golosov dynamic localization ([Fisher-Le Doussal-Monthus])

Random potential (infinite dimension)

- One particle in a Gaussian random potential $V(\mathbf{r})$ in $N \to \infty$ dimensions:

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = Ng^2 F[\frac{|\mathbf{r}-\mathbf{r}'|^2}{2N}],$$

- Short range: $V_{\text{max}}(R) \sim \sqrt{N} g \sqrt{2 \ln R^N} \sim N g \sqrt{2 \ln R}$ can compete with $S \sim N \ln R$ at fixed R but $N \to \infty \to$ true phase transition

- Exactly soluble model in the large N limit using replicas ([Mézard-Parisi, Fyodorov-Sommers])

Random potential (infinite dimensions)

- Short range ($H < 0$): 1 step RSB but $T_c(R) \to 0$ for large R
- Long range ($H > 0$): full RSB but $T_c(R) \to \infty$ for large R
- Special case, logarithmically growing F: 1-RSB with *marginally stable* modes for all $T < T_c = g$ (remains finite at large R)

Random potential (log case)

- A logarithmically correlated random potential in N dimensions:

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = N\left[f_0 - g^2 \ln(|\mathbf{r}-\mathbf{r}'|^2 + a^2)\right]$$

- Simple argument: $V_{\text{max}} \sim g\sqrt{N \ln R}\sqrt{\ln R^N} \sim Ng \ln R$ matches S at $T_c = g$

- Free energy given by a REM-like expression with a freezing transition at $T_c = g$ indep of N, using RG methods ([Carpentier-Le Doussal]) – matches exact results at $N = \infty$ ([Fyodorov-Sommers])

- Low-energy states still have a Gumbel-like distribution (with pre-exponential corrections)

Random potential (log case)

- Interesting dynamics: $r^2(t) \sim t^{2/z}$ with a g dependent exponent $z = 2+2(g/T)^2$, and a dynamical transition at T_c, where z becomes $4g/T$ ([Castillo-Le Doussal]) and *aging* sets in

- Building block of the Bacry-Muzy-Delour multifractal random walk (finance):

$dX(t) = \sigma(t)dW(t) \qquad \sigma(t) = \sigma_0 \exp[-\beta V(t)] \qquad V$ Gaussian

$$\langle (X_t - X_{t+\tau})^n \rangle = M_n \tau^{\zeta_n} \quad \text{with} \quad \zeta_n = \frac{n}{2} - (\beta g)^2 \frac{n(n-2)}{2},$$

A multiscale logarithmic potential

- Logarithmically correlated random potential with several scales:
  $V(\mathbf{r}) = \sum_{i=1}^{K} V_i(\mathbf{r})$ with

  $$\left\langle V_i\left(\mathbf{r}_1\right)\, V_j\left(\mathbf{r}_2\right)\right\rangle_V = \delta_{i,j} N\, F_i\left(\frac{1}{2N}(\mathbf{r}_1 - \mathbf{r}_2)^2\right), \qquad f_i(u) = -g_i^2 \ln\left(u + R^{2\nu_i}\right)$$

  with increasing $0 \leq \nu_i \leq 1$ – separation of length-scales in the $R \to \infty$ limit

- In the continuum limit $g^2 \ln(r^2+a^2) \to \int_0^1 \rho(\nu) g^2(\nu) \ln\left(r^2 + R^{2\nu}\right) d\nu$,

  $$F(x) = -\ln R\,\Phi\left(\frac{\ln x}{\ln R}\right) \qquad \Phi(y) = y\int_0^y \rho(\nu) g^2(\nu)\, d\nu + \int_y^1 \nu\rho(\nu) g^2(\nu)\, d\nu,$$

- Exact results for the free-energy $\mathcal{F}$ in the $N \to \infty$ limit for any Φ ([Fyodorov-JPB])

A multiscale logarithmic potential

- For a discrete spectrum $g^2(\nu)\rho(\nu) = \sum_{i=1}^{K} g_i^2\,\delta(\nu - \nu_i)$, the model has exactly the GREM free energy in the limit $R \to \infty$.

- For each temperature $T_p = \sum_{i=p}^{K} g_i^2$ the system freezes within blobs of size R^{ν_p} *à la* REM, with a Parisi $x = T/T_p$

- The system first freezes at $T_c = T_1$ on the largest scale

- The last freezing transition takes place at $T_{\text{min}} = T_K$ on scale R^0

- Each freezing transition is characterized by a participation ratio $Y_2(R^{\nu_p}) = 1 - \frac{T}{T_p}$

A multiscale logarithmic potential

- In the continuum limit: an explicit construction of Parisi landscapes in finite Euclidean dimensions in terms of GTI processes

- Multifractal Boltzmann measure

$$Y_q = \int_V p_\beta^q(\mathbf{r})\, d\mathbf{r} \sim V^{-\tau_q}$$

  leads to $f(\alpha)$ that are generically singular when $f(\alpha_\pm) = 0$

- Multiscale dynamical freezing

$$z(\ell_p) = 2 + 2\left(\frac{T_p}{T}\right)^2,$$

  all levels such that $T < T_p$ age (concerning large length scales), whereas small length scales, such that $T > T_p$, are still stationary: temperature as a microscope

A multiscale logarithmic potential

- Generalized multifractal random walk with epoch dependent multifractal spectrum (discrete)

$$\zeta_n^{(p)} = \frac{n}{2} - \left(\frac{T_p}{T}\right)^2 \frac{n(n-2)}{2}.$$

- Generalized multifractal random walk (continuum limit)

$$\zeta_n = 2m\beta\mathcal{F}(2m\beta) + m[1 - 2\beta\mathcal{F}(2\beta)] \qquad m = \frac{n}{2} \qquad (\tau \sim R)$$

and $\zeta_n(p)$ with $p = 1 - \log\tau/\log R$ and $T_p = \int_p^1 \rho(\nu) g^2(\nu) d\nu$