Injected power fluctuations in 1D dissipative systems

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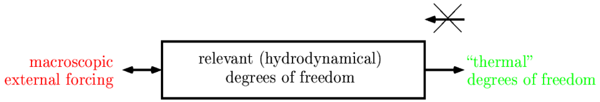
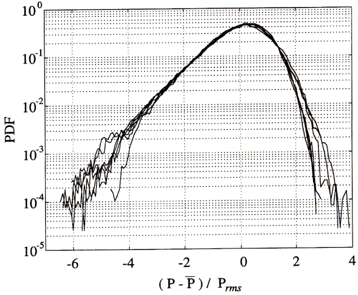
Introduction

- Dissipative stationary states ubiquitous in physics :
  - turbulent stationary flows
  - vibrated granular materials
  - Any system with a “markovian coarse graining”.
- Always structured according to the following scheme :



- No detailed balance, no $t \to -t$ invariance

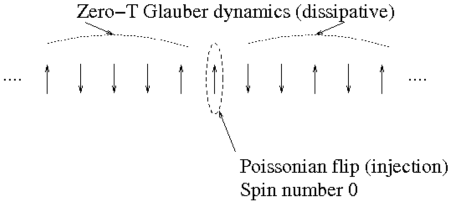
- External forcing fluctuations are of interest for the physicist:
  - Not impossible to measure experimentally (Labbe,Pinton,Fauve 1996):



  - Input for models of turbulence (large-scale forcing)
  - Related to the dissipated power, i.e. to the entropy production.

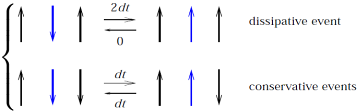
- Is it possible to understand the fluctuation properties of the injected power in a dissipative NESS ? Do exist common features ?
- What is the relation between this coarse-grained approach and the exact microscopic relation, the so-called Fluctuation Theorem ?
- We consider in this talk a (family of) toy-model of dissipative system, driven in a non-trivial stationary state, and perform exact computations related to the injection properties.

A model of dissipative spins



- The zero-T Glauber dynamics is :

individual transition rates



- The spin variables are $s_j = \pm 1$ for $j = -N, N-1$.
- System more easily described by the domain wall variables:

$$n_j = (1 - s_j s_{j+1})/2$$

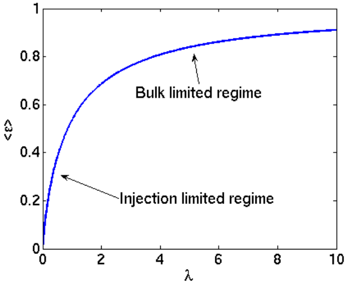
  (also the energy density)
- supresses the trivial symmetry ($\forall j \; s_j \to -s_j$).

- Stationary state characterized by a mean energy profile:

$$\langle n_i \rangle \sim (\pi|i|)^{-1}$$

- And an average injected power

$$\langle \varepsilon \rangle = 2\lambda[\text{Prob}(s_0 = s_1) - \text{Prob}(s_0 = -s_1)]$$
$$= 2\lambda \langle s_0 s_1 \rangle = 2\lambda[\lambda + 1 - \sqrt{\lambda^2 + 2\lambda}]$$



Beyond the mean values

- The instantaneous injected power $\varepsilon(t)$ has a singular pdf : $P(\varepsilon) = A\delta(\varepsilon) + P_{reg}(\varepsilon)$.
- Consider rather

$$\Pi = \int_0^\tau du\varepsilon(u)$$

- What is the distribution of Π/τ for large τ ?
- Large deviation theorem states that

$$\text{Prob}(\Pi/\tau = \varepsilon) \underset{\text{large } \tau}{\propto} \exp\left(\tau f(\varepsilon)\right)$$

$f(\varepsilon)$ is called the large deviation function (ldf)

Properties of the ldf

- The ldf characterizes the fluctuations beyond the central limit theorem (restricted to $f'(\langle\varepsilon\rangle)$ and $f''(\langle\varepsilon\rangle)$).
- General properties: $f(\varepsilon) \leqslant 0$, concave, $f(\langle\varepsilon\rangle) = 0$.
- Time-averaging $\simeq$ low-band filtering, close to experimental measurements.
- But an involved object: the knowledge of the full dynamics is required to compute $f(\varepsilon)$.

The ldf for the spin model

- $f(\varepsilon)$ is given by the inverse Legendre transform of

$$g(\alpha) = \frac{2}{\pi}\int_0^\infty du\ \log\left(1 + \frac{\lambda^2(e^{2\alpha}-1)(\psi_u + e^{2\alpha})}{(\psi_u+1)^2(\lambda^2/4+u^2)}\right)$$

$$\psi_u = \left|2iu + 1 + 2\sqrt{-u^2+iu}\right|^2$$

that is

$$f(\varepsilon) = \min_\alpha (g(\alpha) - \alpha\varepsilon)$$

How to get f ?

- The dynamics of the system is given by a master equation

$$\partial_t P(\mathcal{C}) = \sum_{j=-N}^{N-1} [w(\mathcal{C}_j \to \mathcal{C})P(\mathcal{C}_j) - w(\mathcal{C} \to \mathcal{C}_j)P(\mathcal{C})]$$

$$\mathcal{C} = (s_{-N} = \pm 1, s_{-N+1}, \dots, s_{N-1})$$

$$= (n_{-N} = 0 \text{ or } 1, \dots, n_{N-1})$$

$$\mathcal{C}_j = \mathcal{C} \text{ except } s_j^{\mathcal{C}_j} = -s_j^{\mathcal{C}}$$

- The stochastic operator is a $2^{2N} \times 2^{2N}$ (sparse) matrix. Its elements are :

$$w(\mathcal{C} \to \mathcal{C}_0) = \lambda \text{ (Poisson)}$$

$$w(\mathcal{C} \to \mathcal{C}_j) = n_j + n_{j-1} \quad \text{for } j \neq 0 \quad \text{(Glauber)}$$

- The master equation is useless to compute pdf of time-extended quantities like $\Pi = \int_0^{\tau} du \varepsilon(u) \dots$
- We have to enlarge the description to include the temporal dimension of Π :

$P(\mathcal{C}, \Pi, t)$ is the probability that the system has the configuration $\mathcal{C}$ at t *and* has received an energy Π from the injection in the interval $[0, t]$

- $P(\mathcal{C},\Pi,t)$ obeys a modified master equation (all n_j refer to state $\mathcal{C}$ and $\hat{n}_j = 1 - n_j$):

$$\partial_t P(\mathcal{C},\Pi) = \lambda T_0 + \sum_{j\neq 0} T_j$$

$$\begin{aligned} T_0 &= P(\mathcal{C}_0,\Pi-2)n_{-1}n_0 + P(\mathcal{C}_0,\Pi+2)\hat{n}_{-1}\hat{n}_0 \\ &\quad + P(\mathcal{C}_0,\Pi)[n_{-1}\hat{n}_0 + \hat{n}_{-1}n_0] - P(\mathcal{C},\Pi) \\ T_j &= P(\mathcal{C}_j,\Pi)(\hat{n}_j + \hat{n}_{j-1}) - P(\mathcal{C},\Pi)(n_j + n_{j-1}) \end{aligned}$$

- Usual trick: consider the Laplace transform wrt Π :

$$F(\mathcal{C}) = \sum_{\Pi} P(\mathcal{C},\Pi)e^{\alpha\Pi}$$

- The evolution equation for F is

$$\partial_t F(\mathcal{C}) = \lambda U_0 + \sum_{j\neq 0} U_j$$

$$U_0 = F(\mathcal{C}_0)e^{2\alpha} n_{-1} n_0 + F(\mathcal{C}_0)e^{-2\alpha} \hat{n}_{-1}\hat{n}_0 + F(\mathcal{C}_0)[n_{-1}\hat{n}_0 + \hat{n}_{-1} n_0] - F(\mathcal{C})$$

$$U_j = F(\mathcal{C}_j)(\hat{n}_j + \hat{n}_{j-1}) - F(\mathcal{C})(n_j + n_{j-1})$$

- Typically, $F(\mathcal{C}, t)$ is a sum of exponential (diagonalization of the master operator), whence

$$F(\mathcal{C}, t) \underset{\text{large } t}{\propto} \exp[g(\alpha)t]$$

where $g(\alpha)$ is the largest eigenvalue of $\lambda U_0 + \sum U_j$.

- The Laplace transform of Prob(Π) is given by

$$\langle e^{\alpha\Pi}\rangle = \sum_{\mathcal{C}} F(\mathcal{C}) \underset{\text{large } t}{\propto} \exp[g(\alpha)t]$$

- The inverse Laplace transform yields the cited result:

$$\text{Prob}(\Pi = t\varepsilon) \propto \frac{1}{2i\pi}\int_{0-i\infty}^{0+i\infty} d\alpha \exp(t[\alpha\varepsilon + g(\alpha)])$$
$$\sim \exp\left(t.\ \min_\alpha[\alpha\varepsilon + g(\alpha)]\right)$$

  (It is a min : maximum principle of complex analysis : no inner maximum)

- So, what is the largest eigenvalue of $U_0 + \sum U_j$?

Fermionic diagonalization

- The answer is almost equivalent to the diagonalization of the operator...it is by chance feasible.
- We consider an abstract state space of dimension 2^{4N}, tensorial product of the individual domain walls state spaces. A basis is the collection of vectors

$$|n_{-N}, \dots, n_{N-1}\rangle = |\mathcal{C}\rangle$$

- Consider the vector $|\phi\rangle = \sum_{\mathcal{C}} F(\mathcal{C})|\mathcal{C}\rangle$

Its time evolution is given by

$$\partial_t|\phi\rangle = H|\phi\rangle$$

- H is given by

$$H = \lambda[e^{2\alpha}\sigma_0^-\sigma_{-1}^- + e^{-2\alpha}\sigma_0^+\sigma_{-1}^+ + \sigma_0^+\sigma_{-1}^- + \sigma_0^-\sigma_{-1}^+ - 1]$$
$$+ \sum_{j\neq 0}[2\sigma_{j-1}^+\sigma_j^+ + \sigma_{j-1}^+\sigma_j^+ + \sigma_{j-1}^+\sigma_j^+ - \sigma_{j-1}^+\sigma_j^+ - \sigma_{j-1}^+\sigma_j^+]$$

where, acting on the j-th domain wall state space,

$$\sigma_j^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_j^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- Example :

$$\begin{aligned}\sigma_0^-\sigma_{-1}|\phi\rangle &= \sum_{\mathcal{C}} F(\mathcal{C})\sigma_0\sigma_{-1}|\mathcal{C}\rangle \\ &= \sum_{\mathcal{C}} F(\mathcal{C})\hat{n}_0\hat{n}_{-1}|\mathcal{C}_0\rangle \\ &= \sum_{\mathcal{C}} F(\mathcal{C}_0)n_0 n_{-1}|\mathcal{C}\rangle\end{aligned}$$

the Wigner-Jordan transformation

- Fermionization:

$$
\begin{aligned}
c_{-N} &= \sigma^{+}_{-N} \\
c^{+}_{-N} &= \sigma^{-}_{-N} \\
c_j &= \sigma^{+}_{j}\sigma^{z}_{-N}\sigma^{z}_{-N+1}\cdots\sigma^{z}_{j-1} \\
c^{+}_j &= \sigma^{-}_{j}\sigma^{z}_{-N}\sigma^{z}_{-N+1}\cdots\sigma^{z}_{j-1}
\end{aligned}
$$

where $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *are* truly fermionic operator, i.e.

$$
\begin{aligned}
\{c_i, c_j\} &= 0 \\
\{c^{+}_i, c^{+}_j\} &= 0 \\
\{c^{+}_i, c_j\} &= \delta_{i,j}
\end{aligned}
$$

- Expressed in terms of the c, c^+, the operator H is quadratic :

$$H = \sum_{n,m} \left[c_n^+ A_{nm} c_m + \frac{1}{2} c_n^+ B_{nm} c_m^+ + \frac{1}{2} c_n D_{nm} c_m \right] - \lambda$$

- One shows that H is diagonalizable, i.e. a linear *fermionic* transformation gives

$$H = \sum_q \Lambda_q \left(\xi_q^+ \xi_q - \frac{1}{2} \right) + \frac{1}{2} \mathrm{Tr}(A) - \lambda$$

whence we deduce the eigenvalues of H

- The eigenvalues of H are of the form

$$\frac{1}{2}\sum_q \varepsilon_q \Lambda_q + \frac{1}{2}\mathrm{Tr}(A) - \lambda$$

with $\varepsilon_q = \pm 1$ and Λ_q the positive eigenvalues of

$$M = \begin{pmatrix} A & B \\ D & -A \end{pmatrix}$$

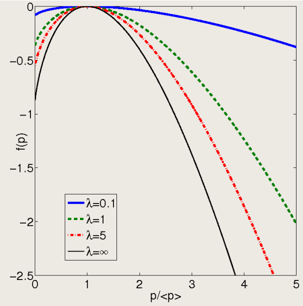
- As a result, we get that

$$g(\alpha) = \frac{1}{4i\pi}\oint d\mu\, \mu \frac{\chi_M'(\mu)}{\chi_M(\mu)} + \frac{1}{2}\mathrm{Tr}(A) - \lambda$$
$$= \frac{2}{\pi}\int_0^\infty du\, \log\left(1 + \frac{\lambda^2(e^{2\alpha}-1)(\psi_u + e^{2\alpha})}{(\psi_u+1)^2(\lambda^2/4 + u^2)}\right)$$

(contour enclosing the positive eigenvalues only)

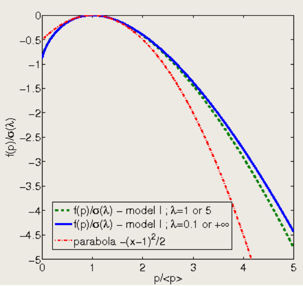
The large deviations

- Typical shape of the large deviation functions (rescaled wrt the mean):



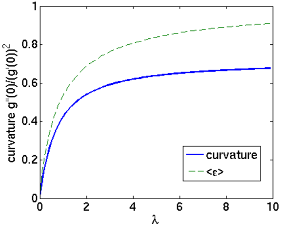
- No negative branch !
- A curvature strongly dependent on λ
- A noticeable positive skewness.

- When rescaled wrt the mean and curvature:

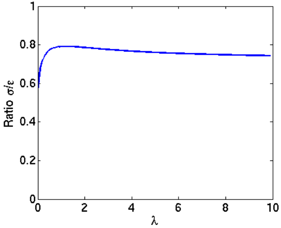


- Skewness not negligible, weak but complicated dependence wrt λ.

The curvature as a function of λ:



$\rightarrow$ almost proportional to $\langle \varepsilon \rangle$:



Why ?

The pure Poissonian model

- Consider an injection of *pairs* of domain walls without negative injection and rate ρ.
- The "injected energy" during τ is simply twice the number n of pairs of dw emitted $\rightarrow$ Poissonian statistics:

$$P(n) = e^{-\rho\tau}\frac{(\rho\tau)^n}{n!}$$

for which one has all the cumulants equal to $\rho\tau$ and

$$\begin{aligned} f(\varepsilon = 2n/\tau) &= \frac{\varepsilon}{2}[1 - \log(\varepsilon/2\rho)] - \rho \\ &= -\frac{\rho}{2}(1-\varepsilon/2\rho)^2 + \frac{\rho}{3!}(1-\varepsilon/2\rho)^3 + \dots \end{aligned}$$

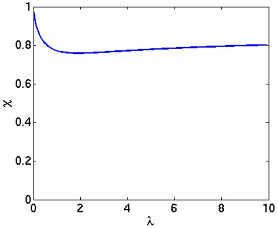
$\rightarrow$ curvature/mean=0.5 here whatever the rate. But we have rather 0.8 except for $\lambda \rightarrow 0\dots$

- A way to improve the model is to assume that in average, $n_{dw}(\lambda)$ are emitted, not necessarily 2
- In that case: curvature/mean=$1/n_{dw}$. Thus, $n_{dw} \sim 1.25$.
- But this model cannot account for the skewness χ, defined by

$$\chi = \langle\langle \varepsilon^3 \rangle\rangle \langle\langle \varepsilon \rangle\rangle / \langle\langle \varepsilon^2 \rangle\rangle^2$$

$$f(\varepsilon) = -\frac{\sigma}{2}(\varepsilon/\langle\varepsilon\rangle - 1)^2 + \frac{\sigma\chi}{3!}(\varepsilon/\langle\varepsilon\rangle - 1)^3 + \ldots$$

- The PPP states $\chi = 1$ whatever n_{dw} whereas we have



Pure Poissonian *discrete* model

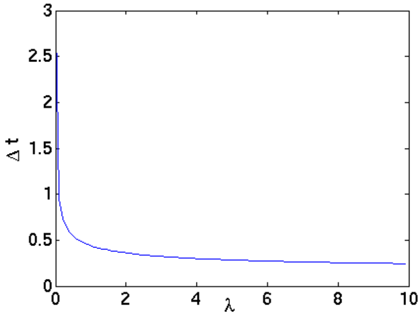
- A slight modification : the emission of n_{dw} domain walls is discrete, with a probability $\rho\Delta t$ in a time interval Δt (remanence).
- In that case,

$$f(\varepsilon) = -\frac{1}{\Delta t}\left[(1-\frac{\varepsilon\Delta t}{n_{dw}})\log\frac{1-\frac{\varepsilon\Delta t}{n_{dw}}}{1-\rho\Delta t} + (\frac{\varepsilon\Delta t}{n_{dw}})\log\frac{\varepsilon\Delta t/n_{dw}}{\rho\Delta t}\right]$$

$$= -\frac{1}{2}(\frac{\varepsilon}{n_{dw}\rho}-1)^2\frac{\rho}{1-\rho\Delta t} + \frac{1}{3!}(\frac{\varepsilon}{n_{dw}\rho}-1)^3\frac{(1-2\rho\Delta t)\rho}{(1-\rho\Delta t)^2} + \ldots$$

- We get $\chi = (1-2\rho\Delta t)/(1-\rho\Delta t)$. This is < 1.
- Conclusion : The domain walls cannot be emitted too closely from each other.

Note: we can extract $\Delta t(\lambda)$ from σ and χ :



$\rightarrow$ not too bad. Defines the time a dw needs to be *effectively* “absorbed” in the system. Related to the dw population near the boundary.

Adding a drift

The model can be generalized in the following way:

- domain walls move toward the boundary with rate $2p < 2$, away from it with rate $2q = 2(1-p)$. Notice $0 < p < 1$
- In that case the Legendre transform of the ldf is

$$\boxed{g(\alpha) = \frac{1}{\pi}\int_{\infty}^{-\infty} du\; u\frac{d}{du}\log F(u) = \frac{2}{\pi}\int_{0}^{\infty} du \log F(u)}$$

$$F(u) = 1 + \lambda^2 \frac{w(u)(w(u)+1)}{\lambda^2/4 + u^2}$$

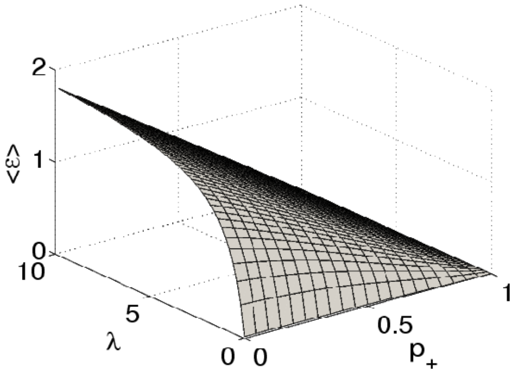
$$w(u) = \frac{2q_+(e^{2\alpha}-1)}{\psi(u) + 4p_+q_+}$$

$$\psi(u) = |x_+(4iu)|^2 = |1 + 2iu + \sqrt{(2iu+1)^2 - \eta^2}|^2$$

- The model has now two parameters : λ and p

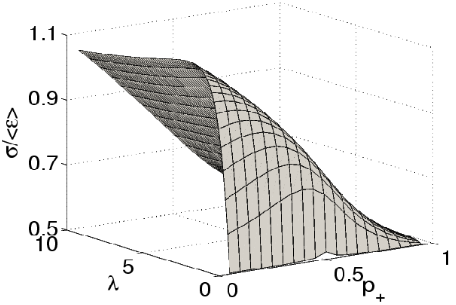
Physical observables

- The mean injected power:



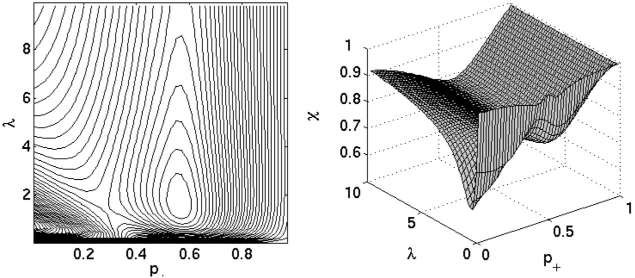
Obviously, the injected power increases with $1 - p$.

- The scaled curvature $\sigma/\langle\varepsilon\rangle$:



- Note a region > 1 near $p = 0$: slightly paradoxical: $n_{dw} < 1$!
- Actually, everything is normalized by the average $\langle\varepsilon\rangle$: n_{dw} measures the decorrelation of the two injections in the two halves of the system.
- For $p = 0$ and $\lambda = \infty$, the two injections are perfectly decorrelated, thus the effective Poisson process emits only one dw.
- Still paradoxical : $p > 1/2$ shouldn't have $n_{dw} = 2$...

The skewness has a complicated behaviour :



→ Still unclear. . .

Conclusion:

- Simple model of energy injection in dissipative systems.
- Quite rich behaviours of the three first cumulants of the distribution.
- Interpretation in terms of effective Poisson processes.
- Can this kind of models be helpful for experiments where dissipative structures are generated near a moving boundary and migrates into the bulk ?
- Perspectives: adding T, just one half of the system, inhomogeneous drift, etc...