1. A BROAD OVERVIEW OF STATISTICAL MECHANICS.

<u>Statistical mechanics</u>: explain macroscopic properties of matter (thermodynamics) in terms of interactions of its microscopic constituents. Also gives probabilistic microscopic description.

<u>Equilibrium stat. mech.</u> (Maxwell, Boltzmann, Gibbs, ...)

$$p_i = \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}} \quad , \quad \beta = \frac{1}{kT}$$

(dynamics can be forgotten)

- entropy $-\sum_i p_i \log p_i$, or $\log N$ (Shannon - McMillan - Breiman thm)
- thermodynamic limit, equivalence of ensembles
- phase transitions, ...

(emergent theory: no time evolution)

NONEQUILIBRIUM SEEN FROM A DYNAMICAL SYSTEMS VIEWPOINT: SRB MEASURES, LINEAR RESPONSE, ETC.

[IHP 10:00 am – noon, 26-28 Sep 2007]

1. A broad overview of statistical mechanics
2. Models of nonequilibrium with NESS
3. Nonequilibrium stat. mech. on phase space M = smooth dynamics on compact manifold M
4. Theory of smooth dynamical systems
5. Linear response

Nonequilibrium stat. mech. (Boltzmann, ...)

- dissipation, irreversibility, entropy production

  (dynamics is important)

- why entropy increases

- why nonequilibrium is difficult:

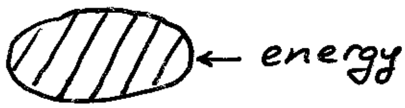
  $\begin{cases} \text{diversity of nonequilibrium processes} \\ \text{necessary use of dynamics} \end{cases}$

  and, far from equilibrium,

  $\begin{cases} \text{entropy not defined} \\ \text{necessity of thermostat} \end{cases}$

Examples

Specific heat

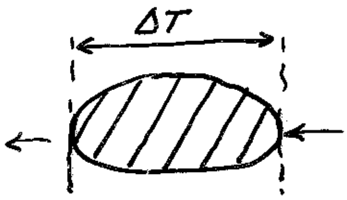


(equilibrium)

easy

raise T
by $1^\circ C$

Heat resistance



(nonequilibrium)

difficult

pass unit energy
per unit time

Chain of harmonically coupled harmonic oscillators doesn't give Fourier's law.

Use chaotic dynamics

$$\delta x(t) \approx \delta x(0)\, e^{\lambda t} \qquad , \quad \lambda > 0$$

Diversity of nonequilibrium

(globular cluster)

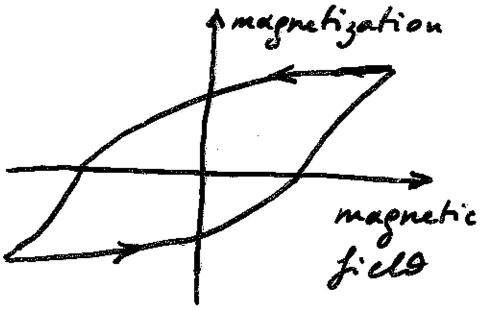
Gravitation

No equilibrium stat. mech. for attracting forces in 3D (or negative temperature electrodynamics in 2D)

(crystal germ)

Metastability

retarded boiling, crystallization, etc.



Hysteresis

The state of the system depends on past history

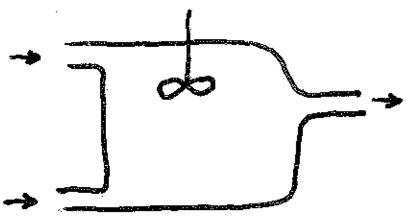
$\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow$

(spin glass)

Glassy systems

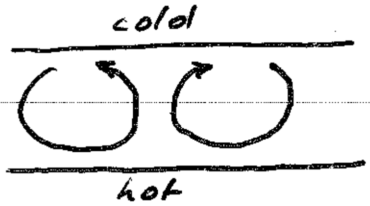
with very long relaxation times

NESS: nonequilibrium steady state



(CSTR)

- chemical reactions



(convection cell)

- transport phenomena (diffusion of solutes, heat, momentum, ...)

Macroscopic evolution equations

- chemical kinetics
  (far from equilibrium)
- transport phenomena:
  choice of variables
  conservation laws
  phenomenological laws
  diffusive character
  transport coefficients $(\ldots c\Delta\ldots)$
  (locally close to equilibrium)
  Onsager, Green-Kubo.

Interlude: Ilya Prigogine

(Jōmon)
against (Boltzmann's) reversible microscopic dynamics

minimum entropy production
chemical kinetics far from equilibrium
dissipative structures

The variety of mathematical idealizations

classical / quantum

stochastic / deterministic

(Boltzman equation, Derrida-Lebowitz-Speer)

[illegible] close to equilibrium / far from equilibrium

(Onsager, Green-Kubo)

infinite thermostat (free / interacting)

/ isokinetic (Gaussian) thermostat

Digressions

Life is far from equilibrium

(but the second law is respected)

The origin of the arrow of time

(how the energy of the sun is used)

Planck's law

energy per unit volume per unit wavelength

$$= \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{kT\lambda}} - 1}$$

Thermostats

Need thermostat except close to equilibrium

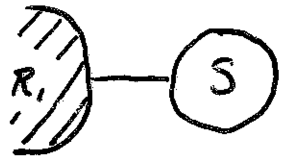
Deterministic thermostat (IK, Gaussian)

works in classical case ($\Rightarrow$ finite system)

not in quantum case

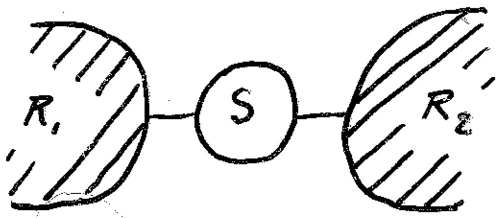
Infinite thermostat (classical or quantum)

isolated system : equilibrium



: approach to equilibrium

or



: NESS (or nonsteady)

Free field thermostat

- photons interact with matter but not with each other
- classical model (Eckmann, Pillet, Rey-B,..) reduces to stochastic b.c.

  entropy creation but Fourier's law not proved

Interacting infinite thermostat

- requires 3D
- must handle dynamics of infinite system: quantum spins, classical ?

2. MODELS OF NONEQUILIBRIUM WITH NESS.

Entropy
Temperature
Local rate of
entropy production

Nonequilibrium
thermodynamics

Non equilibrium
Stat mech

Global rate of
entropy production

Transport
Chemical reactions
Metastability
. . .

away from
equilibrium

close to
equilibrium

Entropy
Temperature

Equilibrium
thermodynamics

Equilibrium
Stat mech

Three Classes of systems

(1) Systems with IK thermostatting

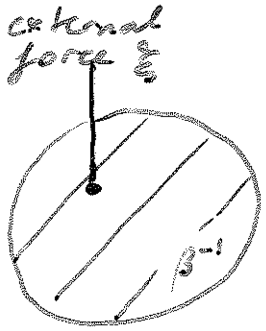
(2) Infinite classical systems of rotators

(3) Infinite quantum spin systems

(1) Systems with IK thermostatting

Case I

$\frac{dx}{dt} = X(x)$



$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \xi(q) - \alpha p \\ p/m \end{pmatrix} \qquad \alpha = \frac{p \cdot \xi(q)}{p^2}$$

$$\Rightarrow \frac{d}{dt}\frac{p^2}{2m} = 0 \qquad K = \frac{p^2}{2m} = \text{constant}$$

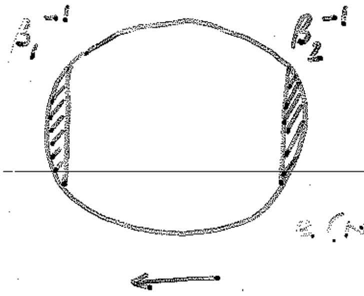
$$S = -\int \rho(x)\, dx \log \rho(x) \,, \quad \frac{dS}{dt} = \int \rho(dx)\, \operatorname{div} X$$

$$-\operatorname{div} X = (N-1)\,\alpha = \beta w$$

$$\text{if} \quad K = (N-1)\frac{\beta^{-1}}{2} \qquad w = \frac{p}{m} \cdot \xi(q)$$

$$e(NESS) = \text{rate of entropy creation} = \beta \int_{NESS} w \geq 0$$

Case II



$$-\operatorname{div} X = \beta_1 w_1 + \beta_2 w_2$$

$$e(NESS) = \text{rate of entropy creation} = (\beta_1 - \beta_2) \int_{NESS} w_1$$

Gallavotti - Cohen Fluctuation Theorem

($\Leftarrow$ Evans, Cohen, Morriss)

We assume that dissipation is present: $e(\rho) > 0$, but the rate $-\mathrm{div}_x X$ fluctuates and can take negative values.

The entropy production over time τ divided by $e(\rho)\tau$, also fluctuates, and we denote its probability distribution by $P^{\tau}(\varepsilon)\, d\varepsilon$.

$$FT \Rightarrow \quad \lim_{\tau\to\infty} \frac{1}{\varepsilon\, e(\rho)\tau} \log \frac{P^{\tau}(\varepsilon)}{P^{\tau}(-\varepsilon)} = 1$$

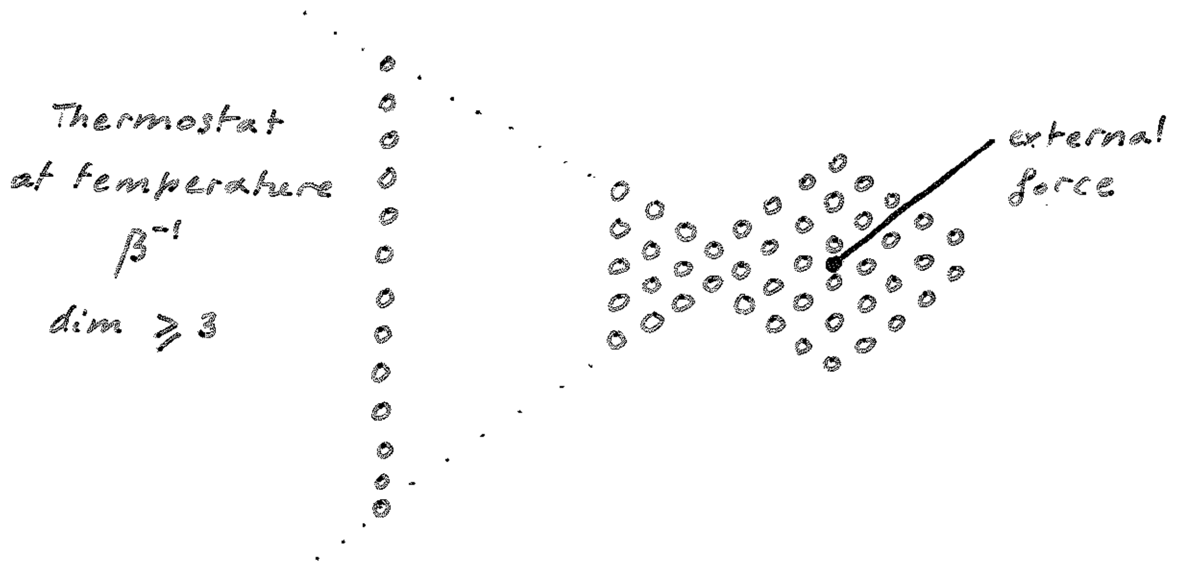
No adjustable parameter

verified in numerical experiments

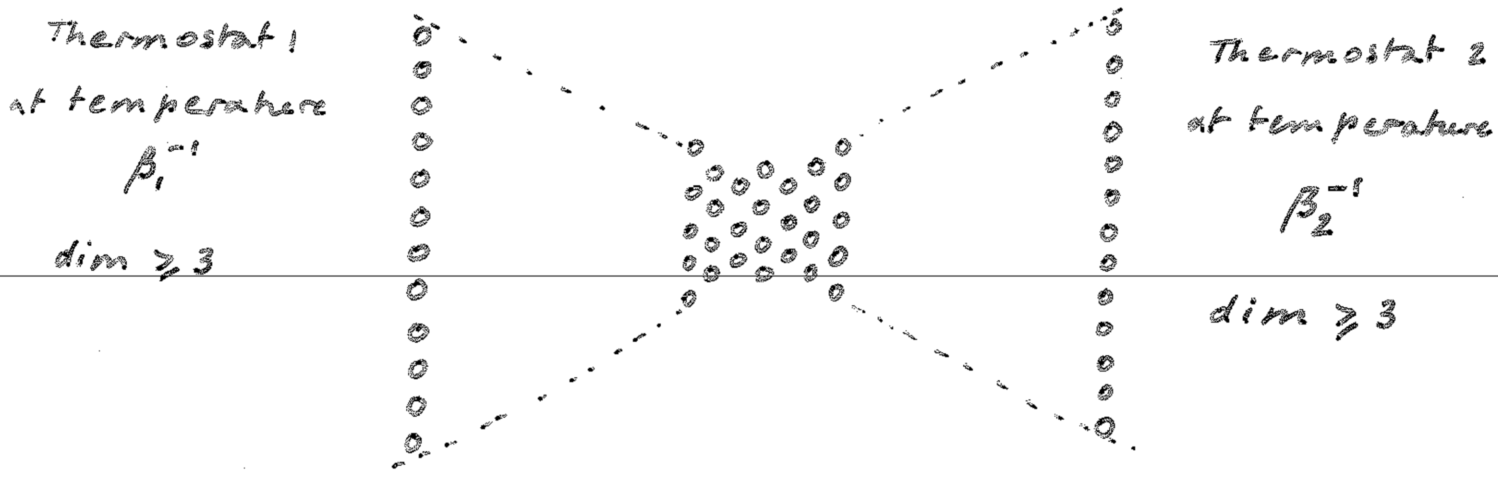
Rigorously proved from chaotic hypothesis (i.e. for Anosov systems).

(1) Infinite classical system of rotators.

Case I



Case II.



$\beta_1^{-1} < \beta_2^{-1}$

$\longleftarrow$

flow of
energy

L : countable set , Σ : graph with vertex set L

$$H_x = \frac{p_x^2}{2m} + U_x(q_x) \qquad \text{if} \qquad x \in L$$

$$H_\Lambda = \sum_{x\in\Lambda} H_x + \sum_{\{x,y\}\in\Gamma_\Lambda} W_{\{x,y\}}(q_x, q_y)$$

Assume : Γ finite dimensional

U_x, $W_{\{x,y\}}$ and derivatives uniformly bd.

- Good time evolution f^t for ∞ Hamiltonian system (possibly with external force).
- Initial state : Gibbs state

$$\ell = \lim_{\Lambda\to\infty} Z_\Lambda^{-1} e^{-\tilde{H}_\Lambda}$$

$$\tilde{H}_\Lambda = \sum_{x\in\Lambda}\left(\frac{p_x^2}{2m} + \tilde{U}_x(q_x)\right) + \sum_{\{x,y\}\in\Gamma_\Lambda} \tilde{W}_{\{x,y\}}(q_x, q_y)$$

- Projection of $f^t\ell$ to region X :

$$\ell_X^t(p_X, q_X)\, dp_X\, dq_X$$

- Entropy $\qquad S^t(X) = -\int dp_X dq_X\, \ell_X^t(p_X, q_X) \log \ell_X^t(p_X, q_X)$

Conditional entropy $\qquad \check{S}^t(X) = \lim_{\Lambda\to\infty} \left(S^t(\Lambda) - S^t(\Lambda\setminus X)\right)$

- NESS $\qquad \rho = \lim_{T\to+\infty} \frac{1}{T}\int_0^T dt\, f^t\ell$
- Entropy production

$$e(X) = -\lim_{T\to\infty} \frac{S^T(X) - S^0(X)}{T}, \quad \check{e}(X) = -\lim_{T\to\infty} \frac{\check{S}^T(X) - \check{S}^0(X)}{T}$$

Case I

$$0 \le e(X) \le \check{e}(X) \le \beta \times \text{energy flux from } \xi$$

Case II

$$0 \le e(X) \le \check{e}(X) \le (\beta_1 - \beta_2) \times \text{energy flux to } 1$$

(3) <u>Infinite quantum spin systems.</u>

L : countable set

If $x \in$ finite $X \subset L$

$\mathcal{H}_x$: finite dim Hilbert space

$\mathcal{H}_X = \bigotimes_{x \in X} \mathcal{H}_x$

$\mathfrak{A}_X$: operators on $\mathcal{H}_X$

$\mathfrak{A}$ = norm closure of $\bigcup_X \mathfrak{A}_X$

$\Phi(X)$: self adjoint $\in \mathfrak{A}_X$

Formal Hamiltonian $H = \sum_X \Phi(X)$

defines time evolution α^t on $\mathfrak{A}$ (Robinson).

σ : Case II initial state $\beta_1^{-1} < \beta_2^{-1}$

– NESS : $\lim_{T \to +\infty} \frac{1}{T} \int_0^T dt\, \sigma(\alpha^t A) = \rho(A)$

– Thermodynamic (global) entropy production

$(\beta_1 - \beta_2) \times$ energy flux to 1 ≥ 0

(Ruelle , Jakšić – Pillet)

Physical quantities.

Time evolution and NESS ρ

well-defined for Classe (1) - (3)

ρ singular, ρ_X not singular ?

Temperature

of thermostats : well defined

inside system : not well defined

not restricted to range $[\beta_1^{-1}, \beta_2^{-1}]$ in Case II

($\Rightarrow$ Assumption: bound on energy in X)

Entropy

global entropy not defined

local Gibbs entropy $S_\rho(X)$ may be well-defined

Global entropy production

Case I : $e = \beta \times$ energy flux from $\xi \geq 0$

Case II : $e = (\beta_1 - \beta_2) \times$ energy flux to $1 \geq 0$

(nontrivial !)

Local entropy production

$$e(X) = -\lim_{T\to\infty} \frac{S^T(X) - S^0(X)}{T}, \quad \check{e}(X) = -\lim_{T\to\infty} \frac{\check{S}^T(X) - \check{S}^0(X)}{T}$$

Class (1)

Class (2) ?

Class (3) Trivial !

3. NONEQUILIBRIUM STAT. MECH. ON PHASE SPACE M

= SMOOTH DYNAMICS ON COMPACT MANIFOLD M

(0) Deterministic evolution with IK thermostat

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \xi(q) - \alpha p \\ p/m \end{pmatrix}, \qquad \alpha = \frac{p \cdot \xi}{p \cdot p}$$

$$K = \frac{p^2}{2m} = \text{const.}$$

- reversible
- conformally symplectic (Dettmann - Morriss, Liverani)
  if ξ locally gradient

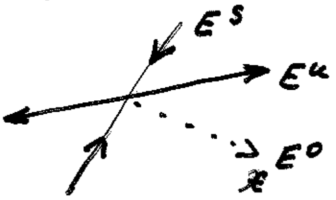
(1) Deterministic time evolution (f^t) on phase space M : differentiable flow or map on compact manifold M

$$\frac{dx}{dt} = \mathcal{X}(x) \qquad\qquad x(t) = f^t x(0)$$

i is a time-reversal symmetry if

$$i^2 = id \quad , \quad i f^t i = f^{-t}$$

(2) <u>Chaotic hypothesis</u> : f^t is uniformly hyperbolic on compact invariant set K



$(\forall t \geq 0)$

$$\| T f^t \xi \| \leq C \theta^{-t} \quad \text{if} \quad \xi \in E^s$$

$$\| T f^{-t} \xi \| \leq C \theta^{-t} \quad \text{if} \quad \xi \in E^u$$

(3) NESS = SRB measure ρ with support in attractor K

$$\rho = \lim_{T\to+\infty} (f^T)^* \text{ normalized Lebesgue}$$

(in basin of K in some sense)

$$= \lim_{T\to+\infty} \frac{1}{T}\int_0^T dt\, \delta_{f^t x} \quad \text{a.e.} \quad (\text{ " " })$$

SRB smooth along unstable directions

$\Leftrightarrow$ $h = \Sigma$ positive Lyapunov exponents

(4) entropy production rate

= volume contraction rate

$$S = -\int m \log m \qquad \frac{dS}{dt} = \int m \operatorname{div} \mathfrak{X}$$

$$e(\rho) = \rho(-\operatorname{div}\mathfrak{X}) = -\Sigma \text{ all Lyap exp of } \rho$$

<u>Consequences</u>

- Gallavotti-Cohen Fluctuation Theorem (Anosov + reversibility)

$$\lim_{\tau\to\infty} \frac{1}{\varepsilon\, e(\rho)\tau} \log \frac{P^\tau(\varepsilon)}{P^\tau(-\varepsilon)} = 1$$

- Linear response

$$\frac{dx}{dt} = \mathfrak{X}(x) + X_t(x), \quad \rho \to \rho + \delta_t\rho$$

$$\int \delta_t\rho(dx)A(x) = \int_{-\infty}^t d\tau \int \rho(dy)\, X_\tau(y)\cdot\nabla_y (A(f^{t-\tau}y))$$

$\Rightarrow$ Green-Kubo formula

Fluctuation-dissipation theorem.

4. THEORY OF SMOOTH DYNAMICAL SYSTEMS

(0) General ergodic theory (abstract or Radon)

- invariant probability measure

every invariant function is constant ρ-a.e.

- ergodic measure ρ $(\rho \neq \alpha \rho_1 + (1-\alpha)\rho_2)$
- Birkhoff pointwise ergodic theorem
- ergodic decomposition of invariant probability measures
  (Bogolinbov-Krylov theory, Choquet theory)
- entropy $h(\rho)$ = KS invariant
- multiplicative ergodic theorem
  Let $(\Omega, \mathbb{P})$ be a probability space
  and $\tau : \Omega \to \Omega$ a map preserving $\mathbb{P}$
  . . . .

(Multiplicative ergodic theorem)

Let $T : \Omega \to M_m$ be a measurable function to real $m \times m$ matrices, such that $\log^+ \|T(\cdot)\| \in L^1(\Omega, \mathbb{P})$. Write $T^n_\omega = T(\tau^{n-1}\omega) \cdots T(\tau\omega) T(\omega)$ and use $*$ to denote matrix transposition.

There is $\Gamma \subset \Omega$ such that $\tau\Gamma \subset \Gamma$, $\mathbb{P}(\Gamma) = 1$, and the following properties hold if $\omega \in \Gamma$:

(a) The limit

$$\lim_{n\to\infty} (T^{n*}_\omega T^n_\omega)^{1/2n} = \Lambda_\omega$$

exists.

(b) Let $\exp \lambda^{(1)}_\omega < \dots < \exp \lambda^{(s)}_\omega$ be the eigenvalues of Λ_ω (where $s = s(\omega)$, the $\lambda^{(r)}_\omega$ are real except that $\lambda^{(1)}_\omega$ may be $-\infty$), and $U^{(1)}_\omega, \dots, U^{(s)}_\omega$ the corresponding eigenspaces. Let $m^{(r)}_\omega = \dim U^{(r)}_\omega$. The functions $\omega \mapsto \lambda^{(r)}_\omega, m^{(r)}_\omega$ are τ-invariant. Writing $V^{(0)}_\omega = \{0\}$ and $V^{(r)}_\omega = U^{(1)}_\omega \oplus \dots \oplus U^{(r)}_\omega$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n_\omega u\| = \lambda^{(r)}_\omega \quad \text{when } u \in V^{(r)}_\omega \setminus V^{(r-1)}_\omega$$

for $r = 1, \dots, s$.

(ergodic case, Lyapunov exponents, multiplicities).

(Invertible case): τ and T invertible

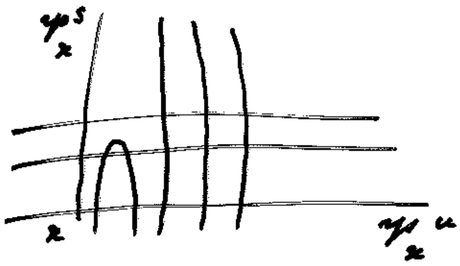
Let $T : \Omega \to GL_m$ be a measurable function to the invertible real $m \times m$ matrices, such that $\log^+ \|T(\cdot)\|, \log^+ \|T^{-1}(\cdot)\| \in L^1(\Omega, \mathbb{P})$. Write $T^n_\omega = T(\tau^{n-1}\omega) \dots T(\tau\omega) T(\omega)$, $T^{-n}_\omega = T^{-1}(\tau^{-n}\omega) \dots T^{-1}(\tau^{-1}\omega)$.

There is then $\Delta \subset \Omega$ such that $\tau \Delta = \Delta$, $\mathbb{P}(\Delta) = 1$, and a measurable splitting $\omega \mapsto W^{(1)}_\omega \oplus \dots \oplus W^{(s)}_\omega$ of $\mathbb{R}^m$ over Δ (with $s = s(\omega)$), such that

$$\lim_{k \to \pm\infty} \frac{1}{k} \log \|T^k_\omega u\| = \lambda^{(r)}_\omega \qquad \text{if } 0 \neq u \in W^{(r)}_\omega$$

(1) Ergodic theory of smooth dynamical systems: ergodic measure ρ on compact manifold M with (f^t). (Oseledec, Pesin, Ledrappier - Strelcyn - Young).

- Lyapunov exponents and decompositions of tangent space
- Stable and unstable manifolds (local & global)



dim $\mathcal{W}^u_x$ = number of positive Lyapunov exponents (with multiplicity)

- SRB measure ρ for C^2 diffeomorphism f

def : the conditional measures σ_a of ρ with respect to a family (Σ_a) constituted of pieces of (local) unstable manifolds $\mathcal{W}^u$ are absolutely continuous w.r.t. Riemann volume on the $\mathcal{W}^u$.

$\Leftrightarrow$ $h(\rho) = \Sigma$ positive Lyapunov exponents for ρ

[in general $h(\mu) \leq \sum$ positive Lyap. exp. for μ].

If ρ is SRB for the diffeo f, and all Lyapunov exponents are $\neq 0$ (nonuniform hyperbolicity) there is a measurable set $S \subset M$ with Rieman volume > 0 s.t.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} = \rho \quad \text{(vaguely)}$$

for all $x \in S$.

- Absolute continuity of foliations (Pesin)

hyperbolic floc

Smooth disks Σ_1, Σ_2 transverse to $\mathcal{W}^s_{loc}$

The holonomy map $\pi_s : \Sigma_1 \to \Sigma_2$ sends sets of zero Lebesgue (= Riemann) measure to sets of zero measure.

[see p. 302 in C. Bonatti, L. Diaz, M. Viana <u>Dynamics beyond Uniform Hyperbolicity</u> Springer, Berlin, 2005]

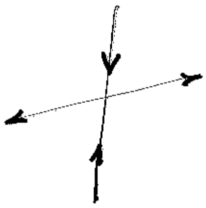
- Problem: noncontinuity w.r.t. f
- Applications to nonequilibrium
  - $e(\rho) = -\Sigma$ all Lyap. exp. of ρ
  - IK + locally gradient

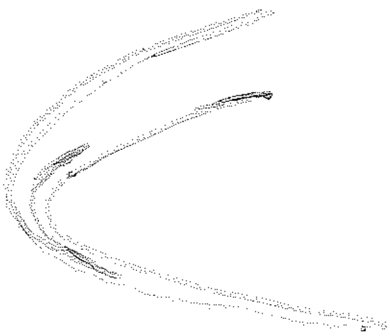
    $\Rightarrow \lambda_i + \lambda_{-i} = \text{const} = \dfrac{e(\rho)}{N-1}$, $\lambda_0 = 0$

    (Dettmann-Morriss pairing theorem)
  - SRB $\Leftrightarrow$

    $h(\rho) = \Sigma$ positive Lyap. exp. of ρ

(2) Various kinds of hyperbolicity
(uniform, non uniform,
maps of the interval, Hénon-like diffeos,
etc.)





(3) Uniformly hyperbolic dynamical systems

- Hyperbolic invariant set K:

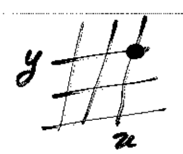
$$T_K M = E^u + E^s \,(+E^0) : \text{continuous splitting}$$

$$\left.\begin{array}{l} \|Tf^{-1}|E^u\| \le \lambda \\ \|Tf|E^s\| \le \lambda \end{array}\right\} \text{for some Riemann metric, } \lambda<1$$

$\Rightarrow$ uniformly flat local $W^{u,s}$

$\Rightarrow$ expansiveness (i.e., $(\forall k\in\mathbb{Z}\ d(f^k x, f^k y)\le\varepsilon)$ implies $x=y$)

- (Anosov : M hyperbolic)
- Axiom A : nonwandering set Ω hyperbolic + periodic points dense in Ω
- Smale's spectral decomposition theorem: Ω is disjoint finite union of basic sets Λ_i (Λ_i compact invariant transitive, locally maximal)

  $\to$ mixing pieces for some f^N
- Axiom A attractor : basic set Λ with open basin $\Leftrightarrow$ $W^u_x \subset \Lambda$ when $x\in\Lambda$
- (structural stability)
- local product structure 

$[x,y]\in\Omega$

  Axiom A $\Rightarrow$ Ω has local product structure

  shadowing (δ-pseudoorbit in Λ is ε-shadowed by true orbit in Λ).

Specification

- <u>Markov partition</u>

Rectangle $R \subset \Lambda$: $x, y \in R \Rightarrow [x,y] \in R$

R closed, $R = \overline{\mathrm{int}\, R}$

Λ has Markov partition of arbitrarily small diameter : finite covering $\{R_1, \dots, R_m\}$ of Λ by rectangles s.t.

$$\mathrm{int}\, R_i \cap \mathrm{int}\, R_j = \emptyset \quad \text{if } i \neq j$$

$$\left.\begin{array}{l} f\, \mathcal{W}^u(x, R_i) \supset \mathcal{W}^u(fx, R_j) \\ f\, \mathcal{W}^s(x, R_i) \subset \mathcal{W}^s(fx, R_j) \end{array}\right\} \text{if } x \in R_i,\ fx \in R_j$$

- <u>Symbolic dynamics</u>

Σ = sequences $(R_{i_k})_{k \in \mathbb{Z}}$ with nearest neighbor condition

$\pi : \Sigma \to \Lambda$

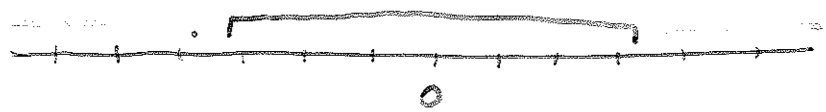
continuous, onto, finite-to-1

invertible on residual set $\bigcap_{n \in \mathbb{Z}} f^n(\bigcup \mathrm{int}\, R_i)$

If σ denotes shift on Σ, then $\pi \circ \sigma = f \circ \pi$

(4) Gibbs states and SRB measures.

- Equilibrium statistical mechanics of 1-dim lattice spin systems



Gibbs state (conditional proba fixed for given interaction)

Equilibrium state ρ (given A = - contribution of one site to total energy, ρ is ergodic measure maximizing

$$h(\rho) + \rho(A)$$

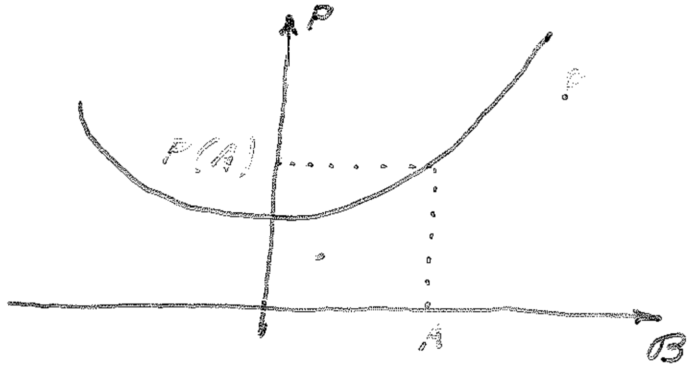
variational principle, the max is $P(A)$)

- In 1-dim for exponentially decreasing interaction

[$\Leftrightarrow$ $A \in \mathcal{B}$: functions on Σ with exponentially small dependence on distant sites

$\Leftarrow$ Hölder functions on Λ]

unique Gibbs state = unique equilibrium state



P real analytic convex

$\rho = \rho_A$ tangent to P at A

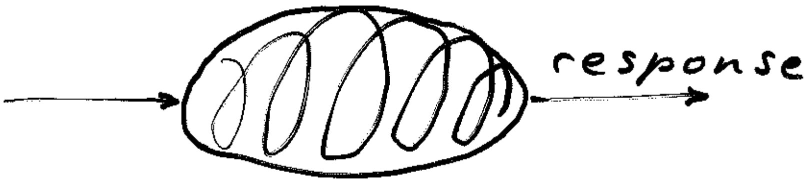
- SRB state on Axiom A attractor is Gibbs state corresponding to $A = -\log J^u$

$$\max\left(h(\rho) - \rho(\log J^u)\right) = 0$$

Some useful books

- C. Bonatti, L. Díaz, M. Viana
  Dynamics beyond Uniform Hyperbolicity
  Springer, 2005
- M. Shub
  Global Stability of Dynamical Systems
  Springer, 1987
- R. Bowen
  Equilibrium states and the Ergodic Theory of Anosov Diffeomorphisms
  LNM 470, Springer 1975
- A. Katok and B. Hasselblatt
  Introduction to the modern theory of dynamical systems
  Cambridge U.P., 1995
- V. Baladi
  Positive Transfer Operators and Decay of Correlations
  World Scientific, Singapore, 2000
- D. Ruelle
  Thermodynamic Formalism
  Addison-Wesley, 1978
  [Cambridge UP 2004]

5. LINEAR RESPONSE



Chaotic dynamical system (f_a^t) with SRB state

$$\rho_a = \lim_{t\to\infty} f_a^t \text{ lebesgue}$$

(Gallavotti - Cohen chaotic hypothesis)

¿ linear response, i.e., differentiability of $a \mapsto \rho_a$?

Nico van Kampen objection

—

Mathematical answer : No

Physical answer : Yes

Green - Kubo formula : transport coefficient

$$= \int dt \ \langle A(0)B(t)\rangle_{eq}$$

= value at $\omega = 0$ of

$$\int_0^\infty dt \ e^{i\omega t} \langle A(0)B(t)\rangle_{eq}$$

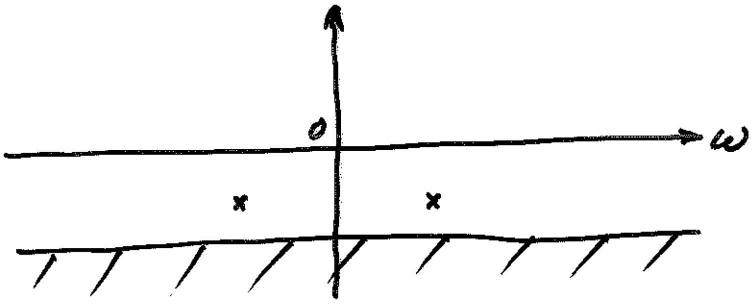
small perturbation (static or periodic)

Chaotic dynamical system
SRB measure ρ
[system outside eq.
NESS ρ]

linear response
[susceptibility (ω)]

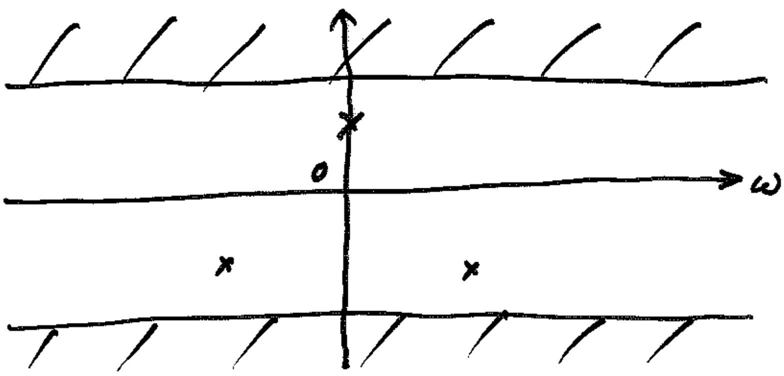
<u>What one expects</u>

If ρ = equilibrium, susceptibility related to Fourier transform of time correlation function (power spectrum)



If ρ not equilibrium : ?

apparent "violation of causality"



[Numerical work needed,

cf Cessac & Sepulchre].

Formal calculation

$$f \mapsto \rho \qquad (\text{a.c.i.m. or SRB})$$

for smooth dynamical system $M \circlearrowleft f$

$$\rho = \lim_{n\to\infty} f^n m \qquad \left(\begin{array}{c} m : \text{normalized} \\ \text{Lebesgue} \end{array}\right)$$

Perturb f by vector field X : $X(fx) = \delta f x$

For A smooth, to 1st order, formally

$$\rho(A) + \delta\rho(A) = \lim_{n\to\infty} \int m(dx)\, A\left((f+\delta f)^n x\right)$$

(*) $$\delta\rho(A) = \sum_{k=0}^{\infty} \int \rho(dx)\, X(x) . \nabla_x (A \circ f^k)$$

or, for time-periodic perturbation, $\lambda = e^{i\omega}$,

$$\Psi(\lambda) = \sum_{k=0}^{\infty} \lambda^k \int \rho(dx)\, X(x) . \nabla_x (A \circ f^k)$$

The function $\omega \mapsto \Psi(e^{i\omega})$: susceptibility

$$\delta\rho(A) = \Psi(1)$$

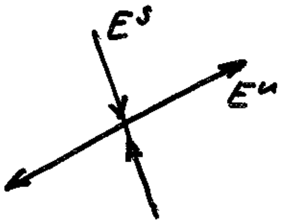
Equilibrium : $\rho(x) = dx$

$$\Psi(\lambda) = - \sum_{k=0}^{\infty} \lambda^k \int dx\, (\operatorname{div} X)(x)\, A(f^k x)$$

$\Rightarrow$ Green - Kubo

Mathematical concepts : hyperbolicity, SRB

⇒ Uniform hyperbolicity (Anosov, Smale)



$$T_K M = E^u + E^s \; (+ E^0)$$

Tf contracts E^s, Tf^{-1} contracts E^u

For Lebesgue a.e. x in the basin of an Axiom A attractor K

(**) $$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} A(f^n x) = \int \rho(dy)\, A(y)$$

ρ : SRB measure on K

ρ smooth along unstable directions

$h(\rho) = \Sigma$ positive Lyapunov exponents of ρ

⇒ For general diffeo f and ergodic μ

E^u_x, E^s_x defined for μ a.e. x (Oseledec)

W^u_x, W^s_x " (Pesin)

ρ is SRB if equivalent properties (Ledrappier, Strelcyn, Young)

ρ smooth along unstable directions

$h(\rho) = \Sigma$ positive Lyapunov exponents

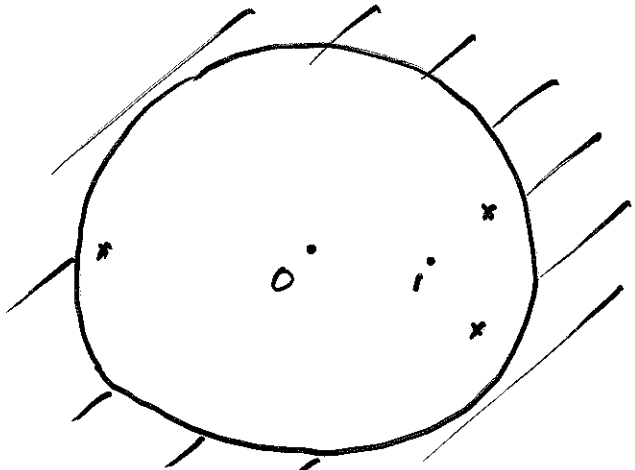
[Then (**) holds for x in a set of positive Lebesgue measure if all Lyap exp $\neq 0$].

Uniformly hyperbolic diffeo f

(Axiom A attractor)

As distribution, SRB state ρ depends differentiably on f; $(*)$ holds for f mixing:
$\delta\rho(A) = \Psi(1)$.

$\Psi(\lambda)$ meromorphic for $|\lambda| < C$ (with $C > 1$)
holomorphic for $|\lambda| \leq 1$



A. Katok, G. Knieper, M. Pollicott and H. Weiss Invent. Math. 98, 581-597 (1989).

[Proof: symbolic dynamics, thermodynamic formalism. Ruelle CMP 187, 227-241 (1997); 234, 185-190 (2003). Other work: Dolgopyat Invent. math 155, 389-449 (2004)]

Calculation : $X = X^s + X^u$

$$\Psi(\lambda) = \sum_{k=0}^{\infty} \lambda^k \int \rho(dx)\, X(x) \cdot \nabla_x (A \circ f^k)$$

$$= \sum_{k=0}^{\infty} \lambda^k \rho\Big(\big\langle (\mathrm{grad}\, A) \circ f^k, (Tf^k) X^s \big\rangle\Big)$$

$$- \sum_{k=0}^{\infty} \lambda^k \rho\Big((A \circ f^k)\, \mathrm{div}^u X^u\Big)$$

Uniformly hyperbolic flow generated by $\mathcal{X}$

(Axiom A attractor)

SRB state ρ depends differentiably on $\mathcal{X}$

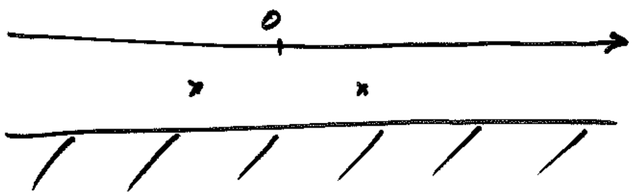
Perturbation X of $\mathcal{X}$

Susceptibility

$$\omega \mapsto \int_0^\infty e^{i\omega t}\,dt \int \rho(dx)\, X(x)\cdot\nabla_x (A\circ f^t)$$

meromorphic for $\operatorname{Im}\omega > -\delta$ (with $\delta > 0$)

holomorphic for $\operatorname{Im}\omega > 0$ (≥ 0 in mixing case)



No pole at 0

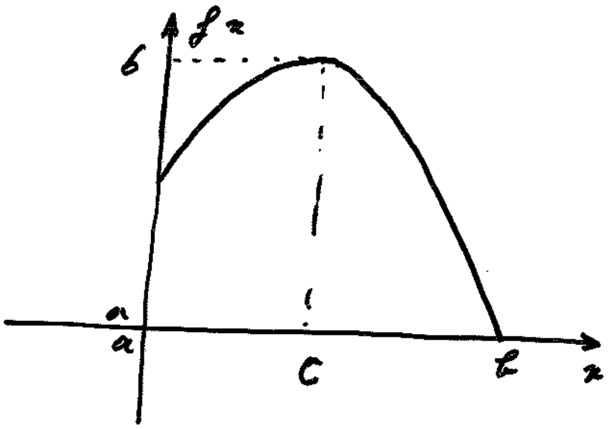
Value at 0 is

$\delta\rho(A)$

[Proof: symbolic dynamics, thermodynamic formalism, transfer operators.

Ruelle ETDS to appear, O. Butterley and C. Liverani to appear].

Unimodal map f of interval I with acim ρ

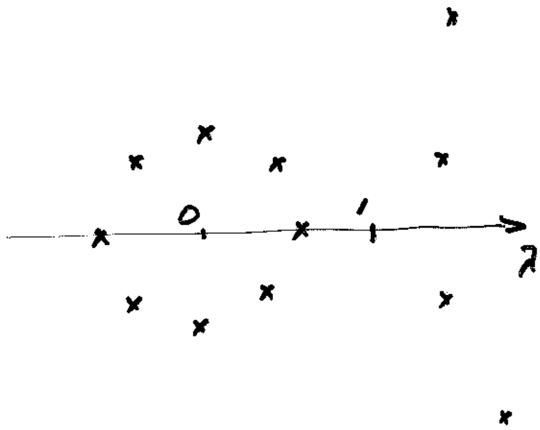


Susceptibility $\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_I \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$

well defined for small λ. What about $\lambda = 1$?

Markovian case (critical point c is preperiodic)

$\Psi(\lambda)$ extends to meromorphic function with poles outside and inside unit circle, but holomorphic at $\lambda = 1$



[Ruelle CMP 258, 445-453 (2005), R. & Y. Jiang Nonlinearity 18, 2447-2453 (2005)]

"violation of causality"

Misiurewicz case (critical orbit captured by hyperbolic invariant set H) $\subset$ Collet-Eckmann

$\xi_\kappa : H \to H_\kappa$ conjugates $f|H$ and $f_\kappa|H_\kappa$, where $f_\kappa = h_\kappa \circ f$, $h_\kappa = id + X$

Topological conjugacy classes

$$f_\kappa^3 c = \xi_\kappa f^3 c$$

X tangent to conjugacy class satisfy horizontality condition

$$X(fc) + \sum_{n=1}^{\infty} \left[\prod_{k=0}^{n-1} f'(f^{k+1}c) \right]^{-1} X(f^{n+1}c) = 0$$

[A. Avila, M. Lyubich and W. de Melo, Invent. Math. 154, 451-550 (2003)].

What appears to be true

[remains to be written down]

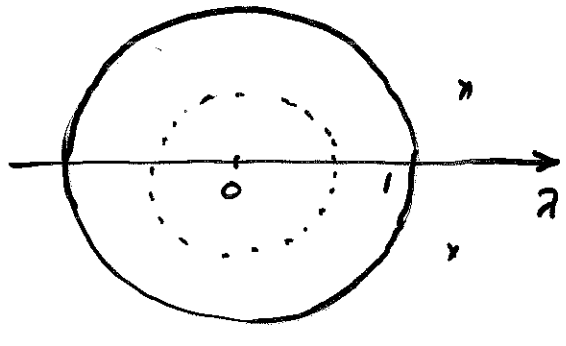
① If X horizontal (codim 1 condition)

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

converges for $|\lambda - 1| < \varepsilon$ and the derivative of $f \mapsto \int A(x) \rho(dx)$ in the X-direction is $\Psi(1)$.

[agrees with conjectures of V. Baladi and D. Smania. To be published].

② If X is nonhorizontal, then $\Psi(\lambda)$ has singularities with $|\lambda| < 1$,



$\kappa \mapsto \int A(x) \rho_\kappa (dx)$ probably not differentiable

"physical derivative $\Psi(\lambda)$" has singularities with $|\lambda| < 1$

(i.e., $\operatorname{Im} \omega > 0$) "violating causality".

References.

- Differentiation of SRB states. Commun. Math. Phys. 187, 227-241 (1997)
- Correction and complements. Commun. Math. Phys. 234, 185-190 (2003)
- Differentiation of SRB states for hyperbolic flows. ETDS, to appear
- Differentiating the absolutely continuous invariant measure of an interval map f with respect to f. Commun. Math. Phys. 258, 445-453 (2005)
- Analyticity of the susceptibility function for unimodal Markovian maps of the interval. (with Yunping Jiang) Nonlinearity, to appear

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- L. Chierchia and G. Gallavotti. Smooth prime integrals for quasi-integrable Hamiltonian systems. Nuovo Cim. 67B, 277-295 (1982)
- J. Pöschel. Integrability of Hamiltonian systems on Cantor sets. Commun. in Pure and Appl. Math. 35, 653-696 (1982)
- H. Whitney. Analytic expansions of differentiable functions defined in closed sets. Trans. Amer. Math. Soc. 36, 63-89 (1934)